

# **Application of Spline Collocation Method to Partial Differential Equation**

A Thesis submitted to Gujarat Technological University

for the Award of

**Doctor of Philosophy**

in

**Science-Maths**

by

**Nileshkumar Arvindbhai Patel**

**[149997673013]**

Under supervision of

**Dr. Jigisha U. Pandya**



**GUJARAT TECHNOLOGICAL UNIVERSITY  
AHMEDABAD**

**December - 2020**

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## List of Symbols

$\rho$	Density of fluid
$u$	Initial Velocity of fluid
$L$	Characteristic linear dimension
$\mu$	Dynamic Viscosity of fluid
$\nu$	Kinematic Velocity of fluid
Re	Reynolds Number
T	Temperature
p	Pressure
A	Area of Cross Section
X	Space Variable
$\Delta$	Difference
$\psi$	Psi
$\phi$	Phi
$\Omega$	Computational domain
$\delta$	Delta
$\varepsilon$	Epsilon
$\sigma$	Sigma
$\xi$	Xi
$\zeta$	Zeta
$\gamma$	Gamma
Y	Space Variable
K	Thermal Conductional
S(x)	Cubic Spline
h	Width of Space Variable
$Q_1$	Quantity of heat
k	Coefficient of Conductivity
S	Specific heat
P	Density of the rod
$\alpha(t)$	Arbitrary function

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### **Future Scope of the work:**

Here, we have examined few applications, which illustrate the usefulness and capacities of the spline explicit and implicit scheme. It is expected that our experience will be helpful in future to obtain solutions to increasingly more linear and nonlinear problems in engineering sciences applied sciences and other fields.

**Objectives:** From the literature survey it is found that, there are many real world problems in which the unknown function involved, depends on several independent variables and gives rise to partial differential equation when they are modeled mathematically. Most of the problems from fluid and solid mechanics, heat transfer, vibrations, electromagnetic theory etc. result into this category. The primary objective of the present work is to determine the successful application of Spline collocation method to solve such equations.

Most of the researchers convert Partial Differential Equations (PDEs) into Ordinary Differential Equations (ODEs) using similarity method and then solve by any numerical method or using available tools like MAPLE, Mathematica, etc. The main objective is to obtain the solution of PDE directly without converting it into ODE to reduce computational complexity using spline collocation method.

The main objective of the work is

- To generalize the Spline collocation method and applying the recurrence relation to find the accurate approximate solution of the fluid flow problems as mentioned above.
- To solve nonlinear coupled Partial differential equations using spline explicit and implicit scheme without converting PDE to ODE.

**Scopes:** The scope of the study involves the following improvements in the method.

- Successfully solve linear one dimensional as well as two dimensional parabolic and hyperbolic partial differential equations using spline explicit and implicit schemes.
- Solution of nonlinear boundary value problems in partial differential equation is a new challenge in flow analysis. Most of the researcher converts partial differential equation in ordinary differential equation using similarity method then solve by any methods, but the new scope is that, without converting into ordinary

differential equation, one can directly solve partial differential equation using different transformations like Hofp-Cole transformation and OST.

- Most of the researchers use numerical method to solve nonlinear partial differential equations problems convert in ordinary differential equations by using similarity method. Due to this it requires more iteration. Hence, generalization of spline explicit and implicit scheme without converting PDE to ODE has been discussed in this work.
- The benefit of this method is, the Spline explicit and implicit scheme can be applied to nonlinear problems based on Hofp-Cole Transformation.
- Couple of equation are solved which can be helpfull to new researchers to solve the coupled nonlinear partial differential equations.
- The applications we have examined are few but sufficient enough to illustrate the effectiveness and potentialities of spline explicit and implicit scheme. It is hoped that others will be able to apply the method with successful results and our experience will help them to obtain solutions to increasingly more difficult and complicated problems in engineering sciences, applied sciences and other fields.
- Our aim is to propose quite a new idea in the theory of splines and that is numerical technique. We hope that our proposed numerical technique will be useful to deal with the non-linear partial differential equations.
- Scopes of the new proposed technique are wide. It will be useful to solve linear as well as nonlinear boundary value problems.

**CHAPTER – 1**  
**A BRIEF DISCUSSION ON DIFFERENTIAL EQUATION**  
**AND SPLINE COLLOCATION METHOD**

- 1.1 Introduction
- 1.2 Classification of Partial Differential Equations
- 1.3 Solution of Partial Differential Equations and Numerical Method
- 1.4 Introduction to Spline Functions
- 1.5 Development of Spline Function
- 1.6 Spline in terms of Moments
- 1.7 Scope of the Present work

## 1.1 INTRODUCTION:

Functions are the basic mathematical tools for describing and analyzing many physical processes of interest. While in some cases these functions are known explicitly, very frequently it is necessary to construct approximations to them based on limited information about the underlying processes. Such approximation problems are a central part of applied mathematics. There are two major categories of approximation problems. The first category consists of problems where it is required to construct an approximation to an unknown function based on some finite amount of data on the function. The second main category of approximation problems arises from mathematical models for various physical processes. As these models usually involve operator equations that determine the unknown function, we refer to them as operator-equation problems. Example includes boundary value problems for ordinary and partial differential equations, and so on.

The differential equations are divided into two classes depending on the number of independent variables present in the differential equation. If the differential equation contains only one independent variable, then it is called ordinary differential equation and if it has more than one independent variable then it is called partial differential equation.

A Partial differential equation describes a relation between unknown functions and its partial derivatives. Partial differential equations appear frequently in all area of physics and engineering. Moreover, in recent years we have seen a dramatic increase in the use of PDEs in areas such as biology, chemistry, computer sciences and in economics. In fact, in each area where there is an interaction between a numbers of independent variables, we attempt to define functions in these variables and to model a variety of processes by constructing equations for these functions. When the value of the unknown functions at a certain point depends only on what happen in the vicinity of this point, we shall, in general, obtain a PDE. The general form of a PDE for a function  $u(x_1, x_2 \dots x_n)$  is

$$F\left(x_1, x_2, \dots, x_n, u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \dots, \frac{\partial^2 u}{\partial x_1 \partial x_n}, \dots\right) \quad (1.1.1)$$

Where  $x_1, x_2 \dots x_n$  are independent variables,  $u$  is the unknown function. The equation is, in general, supplemented by additional conditions such as initial conditions or boundary conditions.

The analysis of PDEs has many facets. The classical approach that dominated the nineteenth century was to develop methods for finding explicit solutions. Because of the immense importance of PDEs in the different branches of physics, every mathematical development

that enabled a solution of a new class of PDEs was accompanied by significant progress in physics. Thus, the method of characteristics invented by Hamilton led to major advances in optics and in analytical mechanics. The Fourier method enabled the solution of heat transfer and wave propagation, and Green's method was instrumental in the development of the theory of electromagnetism. The most dramatic progress in PDEs has been achieved in the last 50 years with the introduction of numerical methods that allow the use of computers to solve PDEs of virtually every kind, in general geometrics and under arbitrary external conditions. Equations with variable coefficients, equations in complicated domains, and nonlinear equations cannot, in general, be solved analytically. We shall therefore have an entirely different approach for solving PDEs. The method applied is based on replacing the continuous variables by discrete variables. Thus the continuum problem represented by the PDE is transformed into a discrete problem in finitely many variables. The discipline of numerical solution of PDEs is rather very recent. The first analysis of a discrete approach to a PDE was presented in 1929 by the German-American mathematician Richard Courant (1888-1972), Kurt Otto Friedrichs (1901-1982), and Hans Lewy (1905-1982) for the special case of the wave equation.

A numerical method provides only an approximation to the exact solution. The main idea of a numerical method is to replace the PDE, formulated for one unknown real valued function, by a discrete equation in finitely many unknowns. This discrete equation is called a numerical scheme.

Thus a PDE is replaced by an algebraic equation. When the original PDE is linear, we obtain, in general, a system of linear algebraic equations. We shall demonstrate below that the accuracy of the solution depends on the number of discrete variables, or, alternatively, on the number of algebraic equations. Therefore, seeking an accurate approximation requires solving large algebraic systems.

There are several techniques for converting a PDE into a discrete problem. The most popular numerical methods are finite difference method (FDM), finite element method (FEM) and Spline Collocation Method. All methods can be used for most problems, including equations with constant or non-constant coefficients, equations in general domains and even nonlinear equations. There is an on-going debate on whether one of the methods is superior to the other. Our view is that the Spline Collocation Method is simpler to describe the problem. The FEM & FDM, on the other hand, is somehow 'deeper' from the mathematical point of view, and is more flexible when solving equations in complex geometrics.

## 1.2 CLASSIFICATION OF PARTIAL DIFFERENTIAL EQUATION:

An equation which involves partial derivatives of an unknown function of two or more independent variables is called a partial differential equation. These equations arise in connection with numerous physical and geometric problems. A partial differential equation can be classified according to various properties.

The order is the order of the highest derivatives that appears in the equation, the dimension is the number of independent variables in the equation, sometimes for initial value problems, dimension refers to the number of “Space” variables while “time” is not counted. An equation is said to be linear if it is linear in the unknown variable and its derivatives. A nonlinear differential equation which is linear in the derivatives of the unknown function is sometimes referred to as quasilinear.

Many physical problems can be represented by a second order PDE in the form:

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + Fu = G$$

or

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \quad (1.2.1)$$

Where A, B, C, D, E, F and G are all functions of x and y.

The classification of PDE at any point  $(x_0, y_0)$  depends on the sign of discriminant defined

as:

$$\Delta(x_0, y_0) = \begin{vmatrix} b & 2a \\ 2c & b \end{vmatrix} = b^2 - 4ac$$

If  $\Delta(x_0, y_0) > 0$ , the equation is hyperbolic.

If  $\Delta(x_0, y_0) = 0$ , the equation is parabolic.

If  $\Delta(x_0, y_0) < 0$ , the equation is elliptic.

A PDE may be of one type at a specific point, and of another type at some other point. The above classification can be written in the following manner:

- Parabolic if  $b^2 - 4ac = 0$
- Elliptic if  $b^2 - 4ac < 0$
- Hyperbolic if  $b^2 - 4ac > 0$

Consider a second – order PDE in three independent variable in the form:

$$a_{xx} \frac{\partial^2 u}{\partial x^2} + a_{yy} \frac{\partial^2 u}{\partial y^2} + a_{zz} \frac{\partial^2 u}{\partial z^2} + 2a_{xy} \frac{\partial^2 u}{\partial x \partial y} + 2a_{yz} \frac{\partial^2 u}{\partial y \partial z} + 2a_{zx} \frac{\partial^2 u}{\partial z \partial x} + b_x \frac{\partial u}{\partial x} + b_y \frac{\partial u}{\partial y} + b_z \frac{\partial u}{\partial z} + cu = 0 \quad (1.2.2)$$

The classification of PDE can be done as

First we define a matrix as:

$$A = \begin{bmatrix} a_{xx} & a_{xy} & a_{xz} \\ a_{xy} & a_{yy} & a_{yz} \\ a_{xz} & a_{yz} & a_{zz} \end{bmatrix}$$

The eigenvalues of matrix A are the roots of characteristic equation of A, i.e.  $|A - \lambda I| = 0$

With the help of A &  $\lambda$  (1.2.2) is classified as:

- If all eigenvalues are non-zero and have same sign, except precisely one of them, then equation is of hyperbolic type.
- If  $|A| = 0$ , i.e., one eigenvalue is zero, then equation is of parabolic type.
- If all eigenvalues are non-zero and have same sign, then equation is of elliptic type.

Equation of the form (1.2.1) classified with respect to the sign of the discriminant

$\Delta = b^2 - 4ac$  in the following way.

If  $\Delta < 0$  at a point in (x, y) plane, the equation is said to be of elliptic type, to be of hyperbolic type when  $\Delta > 0$  at that point and to be of parabolic type when  $\Delta = 0$ .

The particular cases of equation (1.2.1) namely

$$u_{xx} + u_{yy} = 0 \text{ (Laplace equation)}$$

$$u_{xx} - \left(\frac{1}{c^2}\right)u_{tt} = 0 \text{ (one dimensional wave equation)}$$

$$u_{xx} - u_t = 0 \text{ (one dimensional heat equation)}$$

Where u(x, y) are space coordinates and t is the time coordinate. It is easy to see that the Laplace equation is of elliptic type, that the wave equation is of hyperbolic type and that the heat conduction equation is of parabolic type.

A problem represented by PDE is said to be well posed if sufficient number of conditions are prescribed so that its solution can be determined uniquely. These conditions may be prescribed either on the boundary surface or on time. When the condition are prescribed on boundary, these condition are called boundary conditions and when prescribed on time, called initial conditions. Broadly speaking both kinds of these conditions may be referred to as boundary conditions.

The boundary conditions are the classified as:

- Dirichlet conditions

Value of the function u is prescribed on the boundary.

➤ Neumann conditions

Value of normal derivative  $\frac{\partial u}{\partial n}$  is prescribed on the boundary where  $n$  is the direction of the outward normal to the surface.

➤ Mixed conditions

A combination of  $u$  and its normal derivative  $\frac{\partial u}{\partial n}$  is prescribed on the boundary,

$$\text{given as } \alpha \frac{\partial u}{\partial n} + \beta u = \gamma.$$

The boundary conditions are important in the study of partial differential equations. The boundary conditions can also be of different types. Dirichlet condition consists in prescribing only the values of the function on a hyper surface. For Neumann condition, only the values of derivative of the function along the normal to a hyper surface are specified. Cauchy condition define both function value as well as the normal derivative along a hyper surface. These boundary conditions apply to second – order differential equations. For equations of higher order, boundary condition may involve derivatives of higher order.

### **1.3 SOLUTION OF PARTIAL DIFFERENTIAL EQUATION AND NUMERICAL METHOD:**

Here we discuss generally the partial differential equation (PDE) and method for finding the solution of PDE. Most of the problems posed by nature, and which are of interest to physicist and mathematicians, are inherently nonlinear and are usually governed by a single or a system of differential equations. Physical examples of linear systems are relatively rare. Nonlinear equation are difficult to solve and the linear the approximation used to describe them are often tacit admission that the underlying equations cannot be solved. In fact the study of nonlinear systems of differential equations is regarded as a difficult and confusing endeavor. When compared with variety of techniques available in linear system theory, the tour for analysis and design of nonlinear systems are limited to some very special categories. In a sense, nonlinear systems must be treated in to and in their full complexity, and so it is not surprising that there exists no general method for solving them. This is the reason when confronted with a nonlinear differential equation, the first approach usually is to linearize it; in other words, to try to avoid the nonlinear aspects of the problem completely. It is indeed a happy circumstance when a solution can be obtained in this way. But in analyzing the behavior of any physical system, one often encounters the

situation where the linearized model is inadequate or inaccurate; that is the time when study of nonlinear models as such becomes imperative.

Some examples of nonlinear equations are as follows:

- General relativity
- The Burger's equations of fluid dynamics
- The Navier-Stokes equation of fluid dynamics
- Nonlinear Optics

Whilst linear analysis is a mathematical discipline began in the nineteenth century and in the intervening years achieved many spectacular successes throughout sciences, on the other hand the nonlinear equations by virtue of their inherent complexity remains much harder to understand because of their lack of simple superposed solutions consequently, Perturbation, asymptotic and Numerical method (Spline explicit and implicit scheme) are being applied to obtain approximate solutions of these equations.

The class of differential equation is too wide and obtaining their solution is an important task, in the study of such differential equations. There are many methods to find the solutions and they are categorized into three wide groups.

(1) Analytical methods (2) Numerical methods (3) Approximation methods

As a matter of fact, very few differential equations have closed form solutions and even if such solutions exist except for a very simple case, lots of mathematical exercise is required to obtain the solution. In the case where an analytical solution of a partial differential equation is not possible or rather too difficult to obtain, one looks for a numerical to it as a compromise between its accuracy and labor involved in getting a closed form solution.

Numerical calculation is commonplace today in fields where it was virtually unknown before 1950. The high-speed computing machine has made possible the solution of scientific and engineering problems of great complexity. This capability has, in turn, stimulated research in numerical analysis since effective utilization of such devices depends strongly upon the continual advance of research in relevant areas of mathematical analysis.

In modern practice most of the unsolved problem in life science, physical science etc. usually governed by linear and nonlinear partial differential equations, can only be treated by numerical approach. As a consequence, specialist to various fields has devoted increasing attention to numerical as opposed to analytical techniques. In the early days of research in numerical analysis because of restricted capacity of computing machines, the

applications of numerical methods where possible to a limited set of problem. Today the situation is different and the computing devices available now are sufficiently advanced and developed to deal with almost an unlimited range of problem what is really needed is merely the right choice of the effective numerical methods for solving them. Thus, rapidly advancing computers have greatly extended the reality of the computational work making it possible in many instances to reject approximate interpretations of applied problem and pass on to the solution of precisely stated problem. No doubt, this involved utilization of a deeper knowledge and understanding of specialized branches of mathematics. Also, the proper aid of modern computer is rather impossible to get without the skilled use of approximation methods and numerical analysis as well. All these need the universally enhanced and inherent in the methods of computational mathematics.

A number of kinds of applications of partial differential equations to nature science have stretched the limit of the field of partial differential equation. It is rather difficult to say that every partial differential equation can be treated numerically by the same method, but it can be modifying by a statement that more than one technique can be attempted to handle the same problem and vice versa. The study of various partial differential equations demand and exclusive range of method for their solutions and several numerical methods are found to be used very frequently, some of these are method of finite differences, finite element method, homotopy perturbation, shooting, and many other methods. It may be added that for some of these methods ready programmed packages are also available for use with modern computers. A numerical solution to a partial differential equation differs in many ways from an analytical solution. The later provides the value of dependent variable corresponding to any value of an independent variable. In contrast to this in numerical solution in interval of interested is divided into predetermined numbers of increments may of equal length. When the equation is to be solved numerically the initial and boundary conditions are necessary to start with at each incremental step. When this solution is completed results are presented through graph and tables. For a sufficiently small step size a numerical solution closely approximates the true once hair grays prematurely, a numerical solution may become unstable. This means that as the solution progresses from one step to next the numerical result may begin to Oscillate in an uncontrolled manner. This referred to as numerical instability. If step size is change then the stability characteristic is varied. The objective is to select an approximate step size so that the numerical solution is reasonably accurate and no instability results or at least the stability is established up to some acceptable limit. Traditionally various choice is in one step size are made and the respective numerical result

are compared. The best solution is then selected. However, the larger step size the shorter is the computer time but it should not be so small to give excessive error. Also, it is desirable that the step size should be small enough to get accurate results.

Numerical Methods are a discipline of mathematics and engineering that came of age with the advent of the computer. Practical systems demanded fast and accurate solutions to complex physical problems, and the computer was developed as the tool to perform these tasks. In fact, a practical engineering situation the problems arise as non-linear differential form, which rarely have exact solutions. Therefore, to overcome such problems, there is a necessity for numerical approximations. In so many problems analytical methods seem to fail to find the solution. Approximate methods or numerical methods are used to solve such equations. The methods in which the solution is approximated by a finite number of terms of some infinite expansions in terms of sequence of functions are known as approximate methods whereas the method in which the solution is approximated numerically at a number of discrete points are known as numerical methods.

The importance of the numerical methods has increased with the development of high-speed digital computers. Fortunately, today's electronic digital computers have expanded the application of numerical techniques. Using these devices, it is a straightforward matter to provide the numerical solution of several thousand – simultaneous linear and non-linear partial differential equations are subjected to a complete set of boundary conditions. This kind of computational speed and accuracy has led to new ideas and concepts for solving large systems of equations.

Computer applications such as MATLAB, Mathematica and Maple showed us the way for the improvement in the numerical methods. These methods provided numerical solutions of partial differential equations during the last five decades as a result, many scientific and engineering problems that involve linear and non-linear partial differential equations, previously unsolved can now be solved by using appropriate numerical methods.

The numerical methods developed may broadly be classified into the following three types: (i) Finite difference methods (ii) Finite element methods (iii) Spline function approximation methods. The case, in which mixed boundary conditions are present in the problem, the spline explicit and implicit schemes give the best solution as compared to finite difference methods and homology methods. In this thesis, the spline function approximation methods to obtain numerical solutions of partial differential equations have been investigated.

## 1.4 INTRODUCTION TO SPLINE FUNCTION:

For many years, long, thin strips of wood or some other material have been used much like French curves by draftsmen to fair in a smooth curve between specified points. These strips or splines are anchored in place by attaching lead weights called ‘ducks’ at points along the spline. By varying the points where the ducks are attached to the spline itself and the position of both spline and the duck relative to the drafting surface, the spline can be made to pass through the specified points provided an enough ducks are used.

The mathematical spline is the result of replacing the draftsman’s spline by its elastic and then approximating the latter by a piecewise cubic with certain discontinuities of derivatives permitted at the junction points where two cubic joins.

The spline approximation in its present form first appeared in a paper by Schoenberg [1946]. After 1946, Schoenberg, together with some of his students, continued these investigations of splines and monosplines. In particular, Schoenberg and Whitney [1949; 1953] first obtained criteria for the existence of certain splines of interpolation. For the case of splines of even order with interpolation at the junction points, a simpler approach to the question of existence due to Ahlberg, Nilson, and Walsh [1964; 1965] is now possible; it makes use of a basic integral relation obtained for cubic splines of interpolation to a function on a mesh  $\Delta$  by Holladay [1957].

In the mathematical spline, we consider an interval  $a \leq x \leq b$ , and subdivide it by a mesh of points corresponding to the locations of the ducks:

The first significant item that one encounters in the extension to polynomial splines is that there is an essential difference between splines of even and odd degree.

Let  $\{x_i\}_{i=0}^n$  be a strictly increasing sequence of points such that  $a = x_0 < x_1 < x_2 < \dots < x_n = b$  consider that  $f(x)$  is a continuous function over an interval  $[a, b]$ . A polynomial spline  $S(x)$  of degree  $m$ , is an interpolate to  $f(x)$  such that

- (i)  $S(x)$  is a polynomial of degree  $m$  over each subinterval  $[x_i, x_{i+1}]$ ,  $i = 0, 1, \dots, n-1$
- (ii)  $S(x_i) = f(x_i)$ ,  $i = 0, 1, 2, \dots, n$
- (iii)  $S(x) \in C^{m-1} [a, b]$

Spline is to split up the domain into different regions and to use a different polynomial for each region. This would be a good strategy if, the function can only be approximated well with a polynomial of a very high order over the entire domain but can be approximated well with a sequence of low-order polynomials for different parts of the domain.

As polynomials, spline functions acquire all the fundamental mathematical attributes. Which means, the spline functions along with their consecutive derivatives are incessant, differentiable and more commonly they are analytic functions.

### **1.5 DEVELOPMENT OF SPLINE FUNCTION:**

The notion of 'Spline' was first mathematically referred to by I.J. Schoenberg in [4], which is probably the first place that the word "spline" is used in connection with smooth, piecewise polynomial approximation. In [2,3], Whitney, a student of Schoenberg working on spline functions, derived a criterion that only certain splines exist for the purpose of interpolation through a given data. The splines of even order interpolating the data at junctions appear in a very simple fashion and their existence criterion was developed by Ahlberg et al in [4, 5, 6, 7]. They also established the best approximation and convergence properties of splines. Almost during the same period Greville in [8], reported that in numerical procedures spline functions can be utilized. In [9], Carl de Boor also established the idea of existence and uniqueness of bicubic splines.

General spline functions with their minimum norm properties were discussed by Lynch et al, Ahlberg et al extended the integral relation to splines in more than one dimensions. The period of 1960 to 1972 was significant in the field of spline theory, as a remarkable research on existence, uniqueness, minimum norm property and best approximation properties was done. During this period Ahlberg et al, Schoenberg, Ziegler, Carl de Boor, Sharma et al, Loscalzo et al, Birkhoff et al, Kershaw, Golombo and many others were actively engaged in the advancement of this theory. In spline function theory, various types of splines are found like generalized splines, B-splines, cardinal splines, Lg-splines, natural splines, polynomial splines, parabolic splines, trigonometric splines. These functions were found as well as studied during the years 1964 to 1969. In Kershaw presented a list of various end conditions mostly occurring in the spline fit problems. In Albasiny et al tried to use spline functions as interpolants of the solution to linear differential equations. The use of B-splines was made to obtain the solution of the equations by Sincovec arising from collocation. In Barrodale et al have shown us how to use standard numerical procedures without a change to calculate the best spline approximation. The spline functions are also useful in the methods developed in mathematical programming.

To obtain the numerical solution to the partial differential equations through spline collocation method is a different approach. The thesis is devoted to study of an application of

spline method along with various features. Special emphasis is given the applicability and reliability of the method of spline collocation. Spline methods are successfully applied to the problems which describe the flow of electricity in the transmission lines, heat conduction in a thin rod, heat flow in thin rectangular plate, finite vibrating string and vibrating membrane. The method is generalized to extend its applications to solve parabolic as well as hyperbolic partial differential equations in one and two space variables. Also, this method is extended to solve more than two spaces, nonlinear parabolic as well as hyperbolic and elliptic partial differential equation. In short for solution this thesis reflects numerical experience with spline function as interpolate to some physical problems, which are of particular interest in the present work. It is well known fact that the integral equations have also become the part of the study of certain physical phenomena. The applications to solve the integral equations are also sought. One can hopefully proceed in this area. These will enlarge the dimensions of the applicability of spline functions.

We shall see that polynomial splines possess the following attractive features:

1. Polynomial spline spaces are finite dimensional linear spaces with very convenient bases;
2. Polynomial splines are relatively smooth functions;
3. Polynomial splines are easy to store, manipulate and evaluate on a digital computer;
4. The derivatives and anti-derivatives of polynomial splines are again polynomial spline whose expansions can be found on a computer;
5. Various matrices arising naturally in the use of splines in approximation theory and numerical analysis have convenient sign and determinantal properties;
6. The sign structure and shape of a polynomial spline can be related to the sign structure of its coefficients.
7. Every continuous function on the interval  $[a,b]$  can be approximated arbitrarily well by polynomial splines with the order  $m$  fixed, provided a sufficient number of knots are allowed;
8. Low-order splines are very flexible, and do not exhibit the oscillations usually associated with polynomials.

## 1.6 SPLINE IN TERMS OF MOMENTS:

While discussing the meaning of mathematical splines it was mentioned that spline theory has an analogy with beam theory in engineering. Hence in the similar fashion, referring to the second derivatives of spline functions as moments Ahlberg et al [1967] represented a cubic spline in terms of its corresponding moments. Use of spline function so expressed to interpolate a function over a given set of data is suggested in this comprehensive collocation of literature about spline theory. Albasiny et al [1971] tried to solve a linear two point boundary value problem by using such type of cubic splines.

Here for the first instance, let us derive this new approach of a cubic spline which is different from that expressed in terms of truncated power function. Consider a linear two point boundary value problem

$$y''(x) + p(x)y'(x) + q(x)y(x) = r(x), \text{ for } a \leq x \leq b \quad (1.6.1)$$

Subject to the boundary conditions

$$G_1[y(a), y'(a)] = 0 \text{ at } x = a \quad (1.6.2)$$

$$G_2[y(b), y'(b)] = 0 \text{ at } x = b \quad (1.6.3)$$

Since  $S(x)$  is a cubic spline interpolating  $Y(x)$  given by equation (1.6.1) we have  $S(x_i) = y(x_i)$  and also  $S''(x)$  is a linear function. Let us define  $S''(x)$  in the subinterval  $[x_i, x_{i+1}]$  of  $[a, b]$  as follows:

$$S''(x) = y_{i+1}'' \frac{x - x_i}{h_{i+1}} + y_i'' \frac{x_{i+1} - x}{h_{i+1}} \quad i = 0, 1, 2, \dots, n-1$$

Where  $h_{i+1} = x_{i+1} - x_i$

Two successive integrations produce a cubic  $S(x)$  in  $[x_i, x_{i+1}]$  which has the form

$$S(x) = y_{i+1}'' \frac{(x - x_i)^3}{6h_{i+1}} + y_i'' \frac{(x_{i+1} - x)^3}{6h_{i+1}} + A \frac{(x_{i+1} - x)}{h_{i+1}} + B \frac{(x - x_i)}{h_{i+1}}$$

With A and B are constants to be determined. Interpolation condition provides their

evaluation. They are  $A = y_i - \frac{h_{i+1}^2}{6} y_i''$  and  $B = y_{i+1} - \frac{h_i^2}{6} y_{i+1}''$

$$S(x) = y_{i+1}'' \frac{(x - x_i)^3}{6h_{i+1}} + y_i'' \frac{(x_{i+1} - x)^3}{6h_{i+1}} + \left( y_i - \frac{h_{i+1}^2}{6} y_i'' \right) \frac{(x_{i+1} - x)}{h_{i+1}} + \left( y_{i+1} - \frac{h_i^2}{6} y_{i+1}'' \right) \frac{(x - x_i)}{h_{i+1}} \quad (1.6.4)$$

Similarly  $S(x)$  can be found in the interval  $[x_{i-1}, x_i]$  as

$$S(x) = y_i'' \frac{(x - x_{i-1})^3}{6h_i} + y_{i-1}'' \frac{(x_i - x)^3}{6h_i} + \left( y_i - \frac{h_i^2}{6} y_i'' \right) \frac{(x - x_{i-1})}{h_i} + \left( y_{i-1} - \frac{h_i^2}{6} y_{i-1}'' \right) \frac{(x_i - x)}{h_i}$$

Deriving  $S'(x)$  at  $x = x_i$  from the (1.6.4) which is denoted by  $S'(x_{i+})$  we have

$$S'(x) = y_{i+1}'' \frac{3(x - x_i)^2}{6h_{i+1}} + y_i'' \frac{(-3)(x_{i+1} - x)^2}{6h_{i+1}} + \left( y_i - \frac{h_{i+1}^2}{6} y_i'' \right) \frac{(-1)}{h_{i+1}} + \left( y_{i+1} - \frac{h_i^2}{6} y_{i+1}'' \right) \frac{(1)}{h_{i+1}}$$

$$S'(x_{i+}) = y_i'' \frac{(-3)(x_{i+1} - x_i)^2}{6h_{i+1}} + \left( y_i - \frac{h_{i+1}^2}{6} y_i'' \right) \frac{(-1)}{h_{i+1}} + \left( y_{i+1} - \frac{h_i^2}{6} y_{i+1}'' \right) \frac{(1)}{h_{i+1}}$$

$$S'(x_{i+}) = -y_i'' \frac{h_{i+1}}{2} - \frac{y_i}{h_{i+1}} + \frac{h_{i+1}}{6} y_i'' + \frac{y_{i+1}}{h_{i+1}} - y_{i+1}'' \frac{h_{i+1}}{6}$$

$$S'(x_{i+}) = -y_i'' \frac{h_{i+1}}{3} + \frac{y_{i+1}}{h_{i+1}} - \frac{y_i}{h_{i+1}} - y_{i+1}'' \frac{h_i^2}{6h_{i+1}} \quad (1.6.5)$$

$$i = 0, 1, 2 \dots n-1$$

And in same way

$$S'(x_{i-}) = y_i'' \frac{h_i}{3} - y_{i-1}'' \frac{h_i}{6} + \frac{y_{i-1}}{h_i} - \frac{y_i}{h_i}, \quad i = 0, 1, 2, \dots, n \quad (1.6.6)$$

Continuity  $S'(x)$  at  $x = x_i$  requires that

$$S'(x_{i+}) = S'(x_{i-})$$

So that

$$-y_i'' \frac{h_{i+1}}{3} + \frac{y_{i+1}}{h_{i+1}} - \frac{y_i}{h_{i+1}} - y_{i+1}'' \frac{h_i^2}{6h_{i+1}} = y_i'' \frac{h_i}{3} + y_{i-1}'' \frac{h_i}{6} - \frac{y_{i-1}}{h_i} + \frac{y_i}{h_i} \quad (1.6.7)$$

This equation gives a system of (n-1) equations in (n+1) variables  $y_i$ ,  $i = 0, 1, 2 \dots n-1$  to be determined. Hence for evaluating these quantities two more conditions are required so that the

system of equations due to (1.6.7) is complete. Ahelberg et al [1967] have shown that the moments  $y_i''$ ,  $i = 0, 1, 2, \dots, n-1$  can be obtained from relation (1.6.7) if a curve is initially fitted to the data and two extra conditions are encountered to complete the system (1.6.7).

Here our objective is to solve a differential equation (1.6.1) with the help of relations (1.6.7).

Let us express the equation (1.6.1) in the form

$$y''(x) = f(x, y, y') \quad (1.6.8)$$

Subject to the boundary condition (1.6.2) and (1.6.3). From these boundary conditions, it is seen that there are four pairs of boundary conditions as possible combinations viz.

- (i)  $y(a) = K$  ;  $y(b) = L$ ,
- (ii)  $y(a) = K$  ;  $y'(b) = L$ ,
- (iii)  $y'(a) = K$  ;  $y(b) = L$                       and
- (iv)  $y'(a) = K$  ;  $y'(b) = L$

The relation will take the form for case (

$$\left. \begin{aligned} -y_1(h_1 + h_2) + h_1 y_2 &= \frac{h_1 h_2}{6} \left( \frac{h_1}{6} y_0'' + \frac{h_1 + h_2}{3} y_i'' + \frac{h_2}{6} y_2'' \right) - h_1 y_0 \\ -(h_1 + h_{i+1})y_i + h_i y_{i+1} &= \frac{h_i h_{i+1}}{6} \left( \frac{h_i}{6} y_{i-1}'' + \frac{h_i + h_{i+1}}{3} y_i'' + \frac{h_{i+1}}{6} y_{i+1}'' \right) - h_{i+1} y_{i-1} \\ -(h_n + h_{n+1})y_{n-1} + h_n y_{n-2} &= \frac{h_n h_{n+1}}{6} \left( \frac{h_n}{6} y_{n-2}'' + \frac{h_{n-1} + h_n}{3} y_{n-1}'' + \frac{h_{n-1}}{6} y_n'' \right) - h_n y_n \end{aligned} \right\} \quad (1.6.9)$$

Case (IV),

$$\left. \begin{aligned} y_0 - y_1 &= -\frac{h_1^2}{6} (2y_0'' + y_i'') - h_1 y_0' \\ h_{i+1} y_{i-1} - (h_i + h_{i+1})y_i + h_i y_{i+1} &= \frac{h_i h_{i+1}}{6} \left( \frac{h_i}{6} y_{i-1}'' + \frac{h_i + h_{i+1}}{3} y_i'' + \frac{h_{i+1}}{6} y_{i+1}'' \right) \\ y_n - y_{n-1} &= \frac{h_n^2}{6} (2y_n'' + y_{n-1}'') + \frac{6}{h_n} y_n' \end{aligned} \right\} \quad (1.6.10)$$

And similarly, we will be able to deal with remaining cases. In these remaining cases either first equation or the last equation of system (1.6.9) or (1.6.10) will be there. It is observed for any case that on the left-hand side the equations are obtained for which the coefficient matrix, for in the matrix form is upper tri-diagonal one.

Now in the process for obtaining a solution to equation (1.6.1) we start with a straight line  $y(x) = mx + c$  passing through the boundary points. Considering this straight line as an initial guess, the moments  $y_i''$  are calculated from the relation (1.6.9) at the nodal points  $x = x_i$  as

$$y_i'' = f(x_i, y_i, y_i'') \quad (1.6.11)$$

For  $i = 0, 1, 2, \dots, n$ . These moments are now utilized to evaluate  $y_i$  through the relation (1.6.9) along with two additional equations sought from boundary conditions (1.6.2) and (1.6.3). This can be furnished by solving merely a tri-diagonal system of equations. The results so obtained can again be improved by continuing the same processes till the desired solutions are found or two successive iterations produce the same values.

Cubic spline approximation of  $y(x)$  given by equation (1.6.1) let us consider that all the nodal points are equal partitioning of the domain  $[a, b]$ , say  $h_{i+1} = h_i = h$  is the length of each partition. In such a situation relation (1.6.9) simplifies to

$$y_{i-1}'' - 2y_i'' + y_{i+1}'' = \frac{h^2}{6} (y_{i-1}'' + 4y_i'' + y_{i+1}'') \quad (1.6.12)$$

Also above equation can be written as

$$S''(x_{i-1}) + 4S''(x_i) + S''(x_{i+1}) = \left( \frac{6}{h^2} \right) (f(x_{i-1}) - 2f(x_i) + f(x_{i+1})) \quad (1.6.13)$$

$i = 1, 2, \dots, n-1$

### 1.7 SCOPE OF THE PRESENT WORK:

This thesis exhibits the study of spline explicit and implicit scheme with different types of approach to solve the linear as well as non-linear boundary value problems. The various features of the Spline explicit and implicit scheme expand the field of numerical solution to the Partial differential equations. A special emphasize is given to the applicability and reliability of the method of spline explicit and implicit scheme, to see this, the spline functions were used to interpolate the solutions to various types of Partial differential equation occurring in the study of several physical phenomena in engineering sciences. In all, it is worthy to mention that the Spline explicit and implicit scheme with all the above approaches are helpful to solve various types of fluid flows problems. The effects of changes in particular parameters in the solution of the problems are shown numerically as well as graphically. We also find the generalization of Spline explicit and implicit scheme by Finite difference technique for one, two and three dimensional problems. The beauty of Spline schemes is that, the partial differential equations can be solved directly without transforming

into ordinary differential form. Coupled equations can also be solved using this method in less number of iterations.

We have restricted our study to the applicability of Spline explicit and implicit scheme for the numerical solution of partial differential equations only. The Spline explicit and implicit scheme can be proved to be at par with the well-known finite difference method and the Finite element method and sometimes better. These are also collocation methods, but Spline explicit and implicit scheme can give more comforts and compactness in the computation. Contemplating the above survey, especially emphasizing the developments and advances in theory of splines, it can be said that spline functions cover many of the approximation aspects. In spite of this, there is a very long way to go and much remains to be done. The application of spline functions in the field of partial differential equations is quite recent topics and needs more analysis from the point of view of existence, uniqueness. This kind of research will certainly explore the field of spline functions more rigorously so that they can be used to solve problems in many areas.

The applications we have examined are few but sufficient enough to illustrate the effectiveness and potentialities of spline explicit and implicit scheme. It is hoped that others will be able to apply the method with successful results and our experience will help them to obtain solutions to increasingly more difficult and complicated problems in engineering sciences, applied sciences and other fields.

## **CHAPTER - 2**

### **APPLICATION OF SPLINE COLLOCATION METHOD TO THE PARABOLIC EQUATION OF ONE SPACE VARIABLE AS WELL AS TWO SPACE VARIABLES**

- 2.1 Spline formula to solve Parabolic linear partial differential equation having One Space Variables
- 2.2 The Heat Conduction Problem
  - 2.2.1 Solution with Spline Explicit Scheme
  - 2.2.2 Solution with Spline Implicit Scheme
  - 2.2.3 Discussion of Result
- 2.3 Spline formula to solve Parabolic partial differential equation with Two space variables
- 2.4 Heat Flow in a Thin Rectangular Plate
  - 2.4.1 Solution with Spline Explicit Scheme
  - 2.4.2 Solution with Spline Implicit Scheme
  - 2.4.3 Discussion of Result
- 2.5 Conclusion

Linear partial differential equations can be solved by various Numerical methods such as finite difference method, method of separable variable, Taylor's series expansion, etc. but sometimes to solve complicated equations much computational work and time is required. This can be made simpler by using Spline Collocation method to solve one and two point boundary value problem. This method reduces the boundary value problem to the problem of solving a set of linear algebraic equations. This Chapter covers a detailed discussion and solution of one and two dimensional parabolic equations using Spline Explicit and Implicit Scheme.

## 2.1 SPLINE FORMULA TO SOLVE PARABOLIC LINEAR PARTIAL DIFFERENTIAL EQUATION HAVING ONE SPACE VARIABLE:

Consider Parabolic PDE having one space variable

$$u_t = c^2 u_{xx} \quad 0 \leq x \leq 1, t \geq 0 \quad (2.1.1)$$

With Dirichlet boundary conditions

$$\begin{aligned} u(0, t) &= 0 \\ u(1, t) &= 0 \end{aligned}$$

Along with initial condition,

$$u(x, 0) = f(x) \quad ; 0 \leq x \leq 1$$

We divide the region  $0 \leq x \leq 1$  into, say  $n$ -equal subinterval each of width  $h$ . Let us denote the points of subdivisions by  $x_0, x_1, x_2, \dots, x_n$ . Let us have the solution on hand at time  $j\Delta t$  at the mesh points  $x_0, x_1, x_2, \dots, x_n$ . Let  $u_{ij}$  denote the value of  $u$  at the  $i^{\text{th}}$  mesh point at time  $j\Delta t$ .

We approximate the function  $u$  at time  $j\Delta t$  by a cubic spline  $S(x)$ : calculate the value of  $S'(x_i)$  by solving set of simultaneous equations. We should note that values of  $u$  at  $x = 0$  and  $x = 1$  are already known. Now discretizing left side of equation (2.1.1) by forward difference formula and replacing right side by the second derivatives  $S''(x_i)$  at  $j^{\text{th}}$  level like explicit scheme in finite difference. We get

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} = c^2 S''_{i,j} \quad (2.1.2)$$

Where  $S''_{i,j}$  denotes  $S''(x_i)$  at  $j^{\text{th}}$  level.

Substitute values of  $S''_{i,j}$  from (2.1.2) into equation (1.6.13) and get

$$u_{i-1,j+1} + 4u_{i,j+1} + u_{i+1,j+1} = (1+6r)u_{i-1,j} + (4-12r)u_{i,j} + (1+6r)u_{i+1,j} \quad (2.1.3)$$

Where  $r = c^2 \Delta t / h^2$ .

This set of simultaneous equations can now be solved equation (2.1.3) known as **cubic spline explicit formula** to solve equation (2.1.1).

The finite difference replacement of corresponding to implicit scheme is

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} = \left( \frac{c^2}{2} \right) (S''_{i,j} + S''_{i,j+1}) \quad (2.1.4)$$

Where  $S''_{i,j}$  and  $S''_{i,j+1}$  denotes second derivatives at  $x = x_i$  at the time level  $j$  and  $j+1$  respectively. We can express (2.1.3) in terms of  $u$  as follows.

$$S''_{i-1,j} + 4S''_{i,j} + S''_{i+1,j} = \left( \frac{6}{h^2} \right) (u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) \quad (2.1.5)$$

$$S''_{i-1,j+1} + 4S''_{i,j+1} + S''_{i+1,j+1} = \left( \frac{6}{h^2} \right) (u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}) \quad (2.1.6)$$

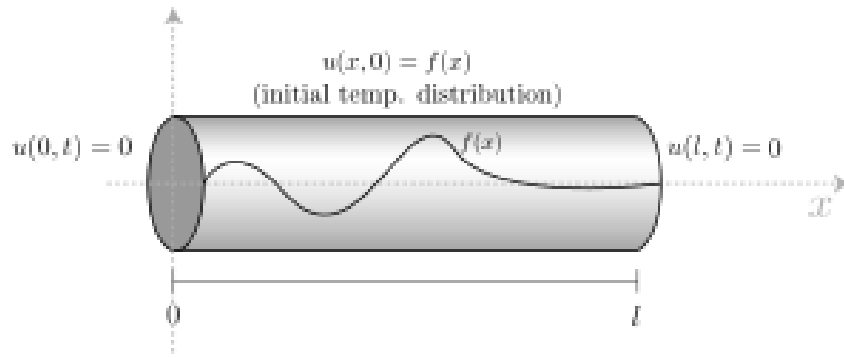
We have added the equation (2.1.5) and (2.1.6) and substitute the value of second derivatives at  $j^{\text{th}}$  and  $(j+1)^{\text{th}}$  level from (2.1.4) to (2.1.5) & (2.1.6), we get

$$(1-3r)u_{i-1,j+1} + (4+6r)u_{i,j} + (1-3r)u_{i+1,j+1} = (1+3r)u_{i-1,j} + (4-6r)u_{i,j} + (1+3r)u_{i+1,j} \quad (2.1.7)$$

Where  $r = \frac{c^2 \Delta t}{h^2}$ .  $i = 1, 2, 3, \dots, n-1$ .

Above equation (2.1.7) is known as **cubic spline implicit formula** to solve equation (2.1.1). Now  $u_{0,j+1}$  and  $u_{n,j+1}$  are known due to the prescribed boundary conditions. The set of simultaneous equations obtained in explicit as well as implicit scheme contains only  $(n-1)$  unknowns. These  $(n-1)$  equations in  $(n-1)$  unknown can be solved by any standard method. The convergence and stability of these methods totally depends upon the values of  $r$ . For explicit scheme, we have to choose the value of  $r$  such that  $0 \leq r \leq 1/2$ . Implicit scheme has advantage that there is no limitation on the value of  $r$  for convergence and stability, even small values are more accurate. Values much larger than unity are not desirable.

## 2.2 THE HEAT CONDUCTION PROBLEM:



Consider a thin long rod surrounded except at the ends with a material impervious to heat unless all the points of the rod are at the same temperature, heat will flow along the rod. If the rod is homogenous and of the same cross section throughout, we may schematically regard the rod as a line, since the temperature of all the points of any cross section will sensibly be the same. When heat is flowing uniformly, it is experimentally known that the amount of heat flow across any portion of the rod is proportional to the difference of temperatures of the end points of the portion, to the area of cross-section and to the end points of the flow and inversely proportional to the length of the portion considered tends to zero, we obtain the quantity of heat  $Q_1$  that flows across any section of the rod as,

$$Q_1 = -k \left[ \frac{\delta u}{\delta x} \right]_x A \quad (2.2.1)$$

Where  $k$  = Coefficient of conductivity

$u$  = The temperature at a distance  $x$  from some fixed point on the rod

$A$  = Area of cross section

The negative been attached as the heat flows from a higher to lower temperature. If we take the section at a point  $x+\delta x$ , then the quantity of a heat that flow out across this section is given by

$$Q_2 = -k \left[ \frac{\delta u}{\delta x} \right]_{x+\delta x} A \quad (2.2.2)$$

Thus from (2.2.1) and (2.2.2) above the quantity of heat gained by this section per second is

$$Q_1 - Q_2 = k \left[ \left[ \frac{\delta u}{\delta x} \right]_{x+\delta x} - \left[ \frac{\delta u}{\delta x} \right]_x \right] A \quad (2.2.3)$$

The rate of rise of temperature is  $\delta u / \delta t$ . Therefore  $Q_1 - Q_2$  is also given by

$$Q_1 - Q_2 = S \cdot P \cdot \delta x \cdot A \cdot \frac{\delta u}{\delta t} \quad (2.2.4)$$

Where  $S$  = Specific heat &  $P$  = Density of the rod

Equating the values of  $Q_1 - Q_2$  from (2.2.3) and (2.2.4) and dividing by  $\delta x$ , we have

$$S \cdot P \cdot \frac{\delta u}{\delta t} = \frac{\left[ \left[ \frac{\delta u}{\delta x} \right]_{x+\delta x} - \left[ \frac{\delta u}{\delta x} \right]_x \right]}{\delta x} \quad (2.2.5)$$

Taking the limit of this equation as  $\delta x \rightarrow 0$ , we get

$$S \cdot P \cdot \frac{\delta u}{\delta t} = k \frac{\delta^2 u}{\delta x^2}$$

$$\frac{\delta u}{\delta t} = \frac{k}{S \cdot P} \frac{\delta^2 u}{\delta x^2}$$

Writing  $a = \frac{k}{S \cdot P}$  the equation of one dimension heat flow is

$$\frac{\delta u}{\delta t} = a \frac{\delta^2 u}{\delta x^2} \quad (2.2.6)$$

In short, we summarize these phenomena by considering a homogenous rod of length  $L$ . The rod is sufficiently thin, so that the heat is disturbed equally over the cross section at time  $t$ . The surface of the rod is insulated and therefore there is no heat loss through the boundary. The temperature distribution of the rod is given by the solution of initial boundary value problem

$$\left. \begin{aligned} u_t &= a u_{xx} & 0 \leq x \leq L, t \geq 0 \\ u(0, t) &= 0 & t \geq 0 \\ u(L, t) &= 0 & t \geq 0 \\ u(x, 0) &= f(x), & 0 \leq x \leq L \end{aligned} \right\} \quad (2.2.7)$$

Where  $u = u(x, t)$  is a function of two variables  $x$  and  $t$ .

- $x$  is the space variable, so  $x \in [0, L]$ , where  $L$  is the length of the rod.
- $t$  is the time variable, so  $t \geq 0$ .

### 2.2.1 SOLUTIONS WITH SPLINE EXPLICIT SCHEME:

We illustrate the problem by considering a homogenous rod of length 1 meter and fluid having unit viscosity( $\nu$ ) and the temperature is given by function  $f(x) = \sin(\pi x)$ , then phenomena can be written as,

$$u_t = a u_{xx} \quad 0 \leq x \leq 1; t \geq 0 \quad (2.2.8)$$

$$\left. \begin{array}{l} u(0, t) = 0 \\ u(1, 0) = 0 \end{array} \right\} t \geq 0 \quad (2.2.9)$$

$$u(x, 0) = x(1-x) \quad 0 \leq x \leq 1 \quad (2.2.10)$$

We shall determine the solution of the equation (2.2.8) satisfying equation (2.2.9) and (2.2.10).

Let us take  $\Delta x = h = 0.1$ ,  $\Delta t = k = 1/500$ ,  $a = 1$  and  $r = 0.2$  which gives

$$1+6r = 1 + 6(0.2) = 2.2 \quad \text{and} \quad 4-12r = 4-12(0.2) = 1.6$$

Substitute the values of  $(1+6r)$  and  $(4-12r)$  in equation (2.1.3) and use initial conditions, we get

For  $i = 1, j = 0$

$$i = 1, \quad u_{0,1} + 4u_{1,1} + u_{2,1} = (2.2)u_{0,0} + (1.6)u_{1,0} + (2.2)u_{2,0} = 0.496$$

$$\text{Since } u_{0,1} = 0, \quad 4u_{1,1} + u_{2,1} = 0.496$$

$$i = 2, \quad u_{1,1} + 4u_{2,1} + u_{3,1} = 0.916$$

$$i = 3, \quad u_{2,1} + 4u_{3,1} + u_{4,1} = 1.216$$

$$i = 4, \quad u_{3,1} + 4u_{4,1} + u_{5,1} = 1.396$$

$$i = 5, \quad u_{4,1} + 4u_{5,1} + u_{6,1} = 1.456$$

$$i = 6, \quad u_{5,1} + 4u_{6,1} + u_{7,1} = 1.396$$

$$i = 7, \quad u_{6,1} + 4u_{7,1} + u_{8,1} = 1.216$$

$$i = 8, \quad u_{7,1} + 4u_{8,1} + u_{9,1} = 0.916$$

$$i = 9, \quad u_{8,1} + 4u_{9,1} + u_{10,1} = 0.496$$

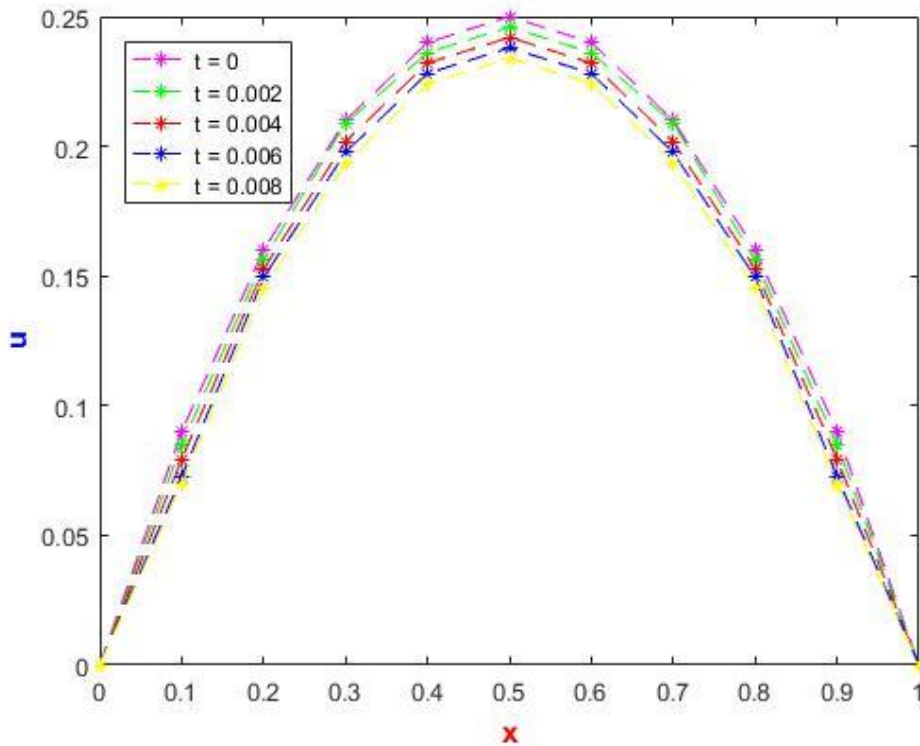
$$u_{10,1} = 0, \quad u_{8,1} + 4u_{9,1} = 0.496$$

Hence we get nine algebraic equations in nine unknowns with tri-diagonal matrix. This system of equation is solved by any well-known method. Similarly, for  $j = 1$ , we get another nine algebraic equations. This can be solved by above method. Proceeding in this way; the results obtained by explicit method are shown in table (2.2.1) and plotted in figure (2.2.1).

**Table: 2.2.1:**

**Temperature in a rod through cubic spline solution by explicit scheme**

Temperature $u(x,t)$				
$x$	$t = 0.002$	$t = 0.004$	$t = 0.006$	$t = 0.008$
0	0.0000000000	0.0000000000	0.0000000000	0.0000000000
0.1	0.0849281767	0.0793204419	0.0726408839	0.0697127071
0.2	0.1562872928	0.1527182320	0.1494364640	0.1451491712
0.3	0.2089226519	0.2018066298	0.1976132596	0.1936906077
0.4	0.2360220994	0.2320552486	0.2281104972	0.2240883977
0.5	0.2459889502	0.2419723756	0.2379447513	0.2339558011
0.6	0.2360220994	0.2320552486	0.2281104972	0.2240883977
0.7	0.2089226519	0.2018066298	0.1976132596	0.1936906077
0.8	0.1562872928	0.1527182320	0.1494364640	0.1451491712
0.9	0.0849281767	0.0793204419	0.0726408839	0.0697127071
1.0	0.0000000000	0.0000000000	0.0000000000	0.0000000000



**Figure: 2.2.1:**

**Temperature in a rod through cubic spline solution by explicit scheme**

### 2.2.2 SOLUTIONS WITH SPLINE IMPLICIT METHOD:

In this section we discuss the solution of (2.2.8) by implicit scheme. Substitute the values of  $r$  and use initial conditions in (2.1.7), we get

For  $i = 1, j = 0$

$$1-3r = 1-3(0.2) = 1-0.6 = 0.4, \quad 1+3r = 1+3(0.2) = 1+0.6 = 1.6,$$

$$4+6r = 4+6(0.2) = 4+1.2 = 5.2, \quad 4-6r = 4-6(0.2) = 2.8$$

$$i = 1, (0.4)u_{0,1} + (5.2)u_{1,1} + (0.4)u_{2,1} = 0.508$$

$$\text{since } u_{0,1} = 0, (5.2)u_{1,1} + (0.4)u_{2,1} = 0.508$$

$$i = 2, (0.4)u_{1,1} + (5.2)u_{2,1} + (0.4)u_{3,1} = 0.928$$

$$i = 3, (0.4)u_{2,1} + (5.2)u_{3,1} + (0.4)u_{4,1} = 1.228$$

$$i = 4, (0.4)u_{3,1} + (5.2)u_{4,1} + (0.4)u_{5,1} = 1.408$$

$$i = 5, (0.4)u_{4,1} + (5.2)u_{5,1} + (0.4)u_{6,1} = 1.468$$

$$i = 6, (0.4)u_{5,1} + (5.2)u_{6,1} + (0.4)u_{7,1} = 1.408$$

$$i = 7, (0.4)u_{6,1} + (5.2)u_{7,1} + (0.4)u_{8,1} = 1.228$$

$$i = 8, (0.4)u_{7,1} + (5.2)u_{8,1} + (0.4)u_{9,1} = 0.928$$

$$i = 9, (0.4)u_{8,1} + (5.2)u_{9,1} + (0.4)u_{10,1} = 0.508$$

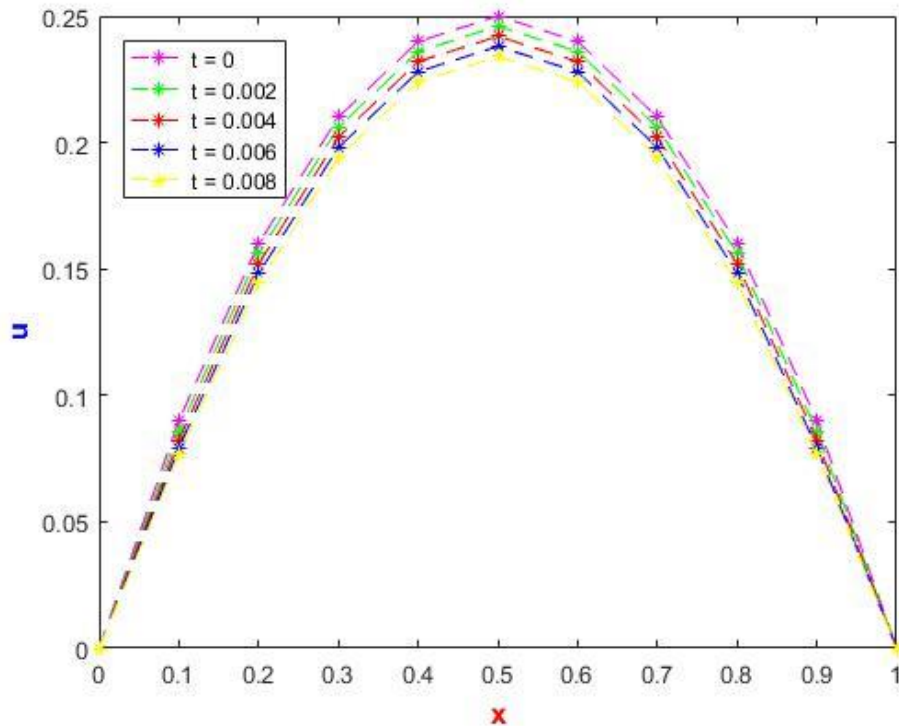
$$u_{10,1} = 0, (0.4)u_{8,1} + (5.2)u_{9,1} = 0.508$$

Hence we get nine algebraic equations in nine unknowns with tri-diagonal matrix. This system of equation is solved by any well-known method. Similarly, for  $j = 1$ , we get another nine algebraic equations. This can be solved by above method. Proceeding in this way; the results obtained by explicit method are shown in table (2.2.2) and plotted in figure (2.2.2).

**Table: 2.2.2:**

**Temperature in a rod through cubic spline solution by implicit scheme**

Temperature $u(x,t)$				
$x$	$t = 0.002$	$t = 0.004$	$t = 0.006$	$t = 0.008$
0	0.0000000000	0.0000000000	0.0000000000	0.0000000000
0.1	0.0856904651	0.0822502446	0.0792774731	0.0766133644
0.2	0.1560239529	0.1520078277	0.1481359949	0.1444268622
0.3	0.2059981463	0.2020002448	0.1980144793	0.1940697699
0.4	0.2360001442	0.2320000076	0.2280015586	0.2240116915
0.5	0.2459999778	0.2420000004	0.2380003281	0.2340037200
0.6	0.2360001442	0.2320000076	0.2280015586	0.2240116915
0.7	0.2059981463	0.2020002448	0.1980144793	0.1940697699
0.8	0.1560239529	0.1520078277	0.1481359949	0.1444268622
0.9	0.0856904651	0.0822502446	0.0792774731	0.0766133644
1.0	0.0000000000	0.0000000000	0.0000000000	0.0000000000



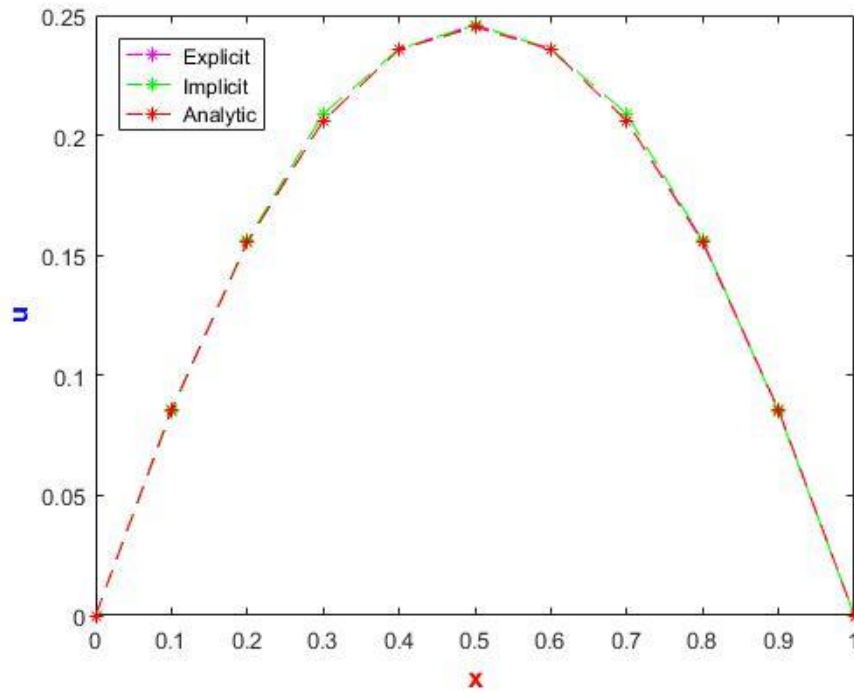
**Figure: 2.2.2:**

**Temperature in a rod through cubic spline solution by implicit scheme**

**Table: 2.2.3:**

**Comparison Spline explicit solution and implicit solution with Analytic Solution**

<b>Temperature <math>u(x,t)</math></b>			
<b>X</b>	<b>Spline Explicit Solution</b>	<b>Spline Implicit Solution</b>	<b>Analytic Solution</b>
0	0.0000000000	0.0000000000	0.000000
0.1	0.0849281767	0.0856904651	0.855324
0.2	0.1562872928	0.1560239529	0.155430
0.3	0.2089226519	0.2059981463	0.206121
0.4	0.2360220994	0.2360001442	0.235764
0.5	0.2459889502	0.2459999778	0.245128
0.6	0.2360220994	0.2360001442	0.235764
0.7	0.2089226519	0.2059981463	0.206121
0.8	0.1562872928	0.1560239529	0.155430
0.9	0.0849281767	0.0856904651	0.855324
1.0	0.0000000000	0.0000000000	0.000000



**Figure: 2.2.3:**

**Comparison Spline explicit solution and implicit solution with Analytic Solution**

### **2.2.3 DISCUSSION OF RESULT:**

Table (2.2.3) gives the comparison of both the solution schemes of spline, namely explicit and implicit with available analytic solutions. From table (2.2.3) it is clear that, the spline solutions are fairly agree with exact solutions, the figure gives good agreement of curves presenting exact and approximate solutions obtained by cubic spline method.

### 2.3 SPLINE FORMULA TO SOLVE PARABOLIC PARTIAL DIFFERENTIAL EQUATION WITH TWO SPACE VARIABLES:

Consider the Parabolic differential equation having two space variables.

$$u_t = c^2(u_{xx} + u_{yy}) ; 0 \leq x \leq a, 0 \leq y \leq b, t \geq 0 \quad (2.3.1)$$

With Dirichlet conditions prescribed on the boundaries  $x = 0$ ,  $x = a$ ,  $y = 0$  and  $y = b$ . we should subdivide the region  $0 \leq x \leq a$  into, say  $M$  intervals, each of width  $\Delta x$  and  $0 \leq y \leq b$  into  $N$  intervals of width  $\Delta y$  such that  $M\Delta x = a$  and  $N\Delta y = b$ . Let us denote the points of subdivisions by  $X_0, X_1, X_2, \dots, X_M$  and  $Y_0, Y_1, Y_2, \dots, Y_N$ . Let  $u_{i,j,k}$  denote the value of  $u$  at the  $(i, j)^{th}$  mesh point at the time  $k\Delta t$ . For simplicity, let us take a square region  $0 \leq x, y \leq a$ ,  $M = N$  and  $\Delta x = \Delta y = h$ .

We approximate the function  $u$  at time  $\Delta t$  by a cubic spline  $S(x)$ . Discretizing the left side of equation (2.3.1) by forward difference formula and replacing right side by twice the second derivative i.e.  $2S''(x_{i,j})$  at  $k^{th}$  level like explicit scheme in finite difference, we get

$$\frac{u_{i,j,k+1} - u_{i,j,k}}{\Delta t} = c^2(S''_{i,j,k}) \quad (2.3.2)$$

Now with the help of equation (1.6.13) and the value of  $S''_{i,j,k}$  obtained from equation (2.3.2) we get,

$$\begin{aligned} \frac{u_{i-1,j,k+1} - u_{i-1,j,k}}{2c^2\Delta t} + \frac{u_{i,j,k+1} - u_{i,j,k}}{2c^2\Delta t} + \frac{u_{i+1,j,k+1} - u_{i+1,j,k}}{2c^2\Delta t} &= \frac{6}{h^2} (u_{i-1,j,k} - 2u_{i,j,k} + u_{i+1,j,k}) \\ u_{i-1,j,k+1} + 4u_{i,j,k+1} + u_{i+1,j,k+1} &= (1+12r)u_{i-1,j,k} + (4-24r)u_{i,j,k} + (1+12r)u_{i+1,j,k} \quad (2.3.3) \\ i, j &= 1, 2, 3, \dots, n-1 \end{aligned}$$

Where  $r = c^2\Delta t/h^2$ .  $i = 1, 2, 3, \dots, n-1$  these set of  $(N-1) \times (N-1)$  equation in  $(N-1) \times (N-1)$  unknowns can be solved by any well-known method. The above set of simulation equation gives square matrix. The equation (2.3.3) is known as cubic spline explicit formula to solve equation (2.3.1). For this method, the maximum possible value of  $r$  is  $1/4$  but the equation (2.3.1) having two space variables and equal grid spacing, hence  $r < 1/6$  is required for stability and convergence. In such case one looks for a method in which  $\Delta t$  can be made larger without loss of stability. Implicit scheme is such a good method in this case.

For the implicit method, the finite difference scheme of equation (2.3.1) is

$$\frac{u_{i,j,k+1} - u_{i,j,k}}{\Delta t} = c^2(S''_{i,j,k} + S''_{i,j,k+1}) \quad (2.3.4)$$

Where  $S''_{i,j,k}$  and  $S''_{i,j,k+1}$  denote second derivatives of  $S(x)$  at  $x = x_{ij}$  at time interval  $k$  and  $k+1$  respectively. We can express (2.3.1) in terms of  $u$  as follows. We use the relationship (1.6.13) and rewrite it as

$$S''_{i-1,j,k} + 4S''_{i,j,k} + S''_{i+1,j,k} = \left(\frac{6}{h^2}\right)(u_{i-1,j,k} - 2u_{i,j,k} + u_{i+1,j,k}) \quad (2.3.5)$$

$$S''_{i-1,j,k+1} + 4S''_{i,j,k+1} + S''_{i+1,j,k+1} = \left(\frac{6}{h^2}\right)(u_{i-1,j,k+1} - 2u_{i,j,k+1} + u_{i+1,j,k+1}) \quad (2.3.6)$$

With the help of equations (2.3.5) and (2.3.6) using the value of  $S''_{i,j,k+1}$  from equation (2.3.4) we get,

$$\begin{aligned} \left\{ \frac{u_{i-1,j,k+1} - u_{i-1,j,k}}{2c^2\Delta t} \right\} - S''_{i-1,j,k} + 4 \left\{ \frac{u_{i,j,k+1} - u_{i,j,k}}{2c^2\Delta t} - S''_{i,j,k} \right\} + \left\{ \frac{u_{i+1,j,k+1} - u_{i+1,j,k}}{2c^2\Delta t} \right\} - S''_{i+1,j,k} \\ = \frac{6}{h^2}(u_{i-1,j,k+1} - 2u_{i,j,k+1} + u_{i+1,j,k+1}) \end{aligned}$$

This gives

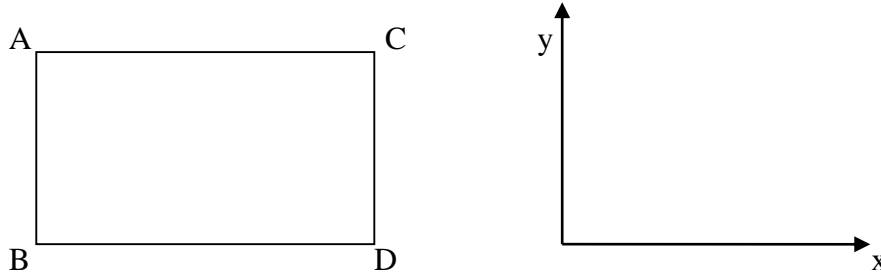
$$\begin{aligned} (1 - 6r)u_{i-1,j,k+1} + (4 + 12r)u_{i,j,k+1} + (1 - 6r)u_{i+1,j,k+1} \\ = (1 + 6r)u_{i-1,j,k} + (4 - 12r)u_{i,j,k} + (1 + 6r)u_{i+1,j,k} \end{aligned} \quad (2.3.7)$$

Where  $r = c^2\Delta t/h^2$ ,  $i = 1, 2, 3, \dots, N-1$  these set of  $(N-1) \times (N-1)$  equation in  $(N-1) \times (N-1)$  unknowns can be solved. The above set of simulation equation gives square matrix. The equation (2.3.7) is known as cubic spline implicit formula to solve equation (2.3.1).

In both methods, once the values of  $u$  are known at  $(k+1)^{\text{th}}$  level, we can proceed to compute the next level  $(k+2)^{\text{th}}$  by same technique as above. The coefficient matrix of the combined equation is a square matrix, however, the system becomes a tri-diagonal one when separate cases are handled. These two methods will be discussed later on by taking its actual approximation to a problem.

## 2.4 HEAT FLOW IN A THIN RECTANGULAR PLATE:

Consider the flow of heat in a thin rectangular plate with sides of length  $\Delta x$  and  $\Delta y$  along co-ordinates  $x$  and  $y$ .



**Figure: 2.4.1: Thin rectangular plate**

The amount of heat entering the element through the side AB in time  $\Delta t$  is

$$-K(\Delta y)(\partial y / \partial x)_x \Delta t$$

And that leaving the element through the opposite CD is

$$-K(\Delta y)(\partial y / \partial x)_{x+\Delta x} \Delta t$$

Where  $K$  is a thermal conductional of the material and  $u(x, y, t)$  is the temperature function. The negative signs are taken because the heat flows in the direction of decreasing temperature. Hence the quantity of heat remaining in the plate as a result of entry through the side AB and exit through the side CD is

$$K \{ (\partial y / \partial x)_{x+\Delta x} - (\partial y / \partial x)_x \} \Delta y \Delta t = K \{ (\partial^2 u / \partial x^2) \Delta x \} \Delta y \Delta t \quad (2.4.1)$$

Up to a first approximation

Similarly corresponding difference in the heat entering and leaving through the remaining pair of opposite side is

$$K \{ (\partial^2 u / \partial x^2) \Delta y \} \Delta x \Delta t \quad (2.4.2)$$

Hence the total heat retained by the plate in time  $\Delta t$  is the sum of results (2.4.1) and (2.4.2), which is equal to the heat required to raise the temperature of the element by  $\Delta u$ .

Thus we have

$$K \{ (\partial^2 u / \partial x^2) + (\partial^2 u / \partial y^2) \} \Delta x \Delta y \Delta t = (\rho \Delta x \Delta y) S \Delta u \quad (2.4.3)$$

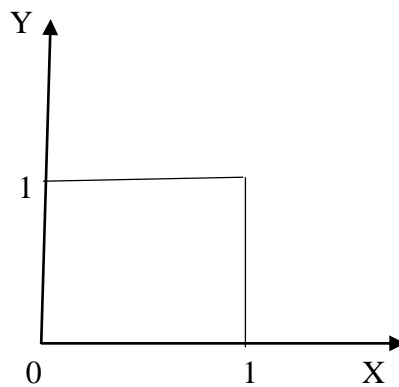
where  $\rho$  is density and  $S$  be the specific heat of the plate. Dividing the equation (2.4.3) by  $\Delta x \Delta y \Delta t S \rho$  and taking limit  $\Delta t \rightarrow 0$ , we get

$$\partial u / \partial t = c^2 \{ \partial^2 u / \partial x^2 + \partial^2 u / \partial y^2 \} \quad (2.4.4)$$

Where  $c^2 = K/S\rho$

Which is parabolic PDE having two space variables  $x$  and  $y$  and time variable  $t$ .

Consider the edges of thin square plate of side length 1 are kept at temperature zero and faces are perfectly insulated.



**Figure: 2.4.2: Thin Square plate**

Hence the flow of heat in the plate is governed by equation (2.4.4) with boundary conditions.

$$\left. \begin{aligned} u(x, 0, t) = 0 \quad 0 \leq x, y \leq 1 \\ u(0, y, t) = 0 \end{aligned} \right\} \quad (2.4.5)$$

And let initial temperature distribution in the plate be

$$u(x, y, 0) = \sin(\pi x) \cdot \sin(\pi y) \quad 0 \leq x, y \leq 1 \quad (2.4.6)$$

### 2.4.1 SOLUTIONS WITH SPLINE EXPLICIT SCHEME:

We shall determine the solution of equation (2.4.4) satisfy boundary conditions (2.4.5) and initial condition (2.4.6) respectively, by using the explicit formula given in equation (2.3.3).

Let  $h = 1/20$ ,  $c^2 = 0.001$  and  $\Delta t = 1/400$  which gives  $r = 0.001$

Hence  $1-6r = 1+12(0.001) = 1.012$  &  $4-24r = 4 - 24(0.001) = 3.976$

Substitute the values of  $1+12r$ ,  $4 - 24r$  with initial and boundary conditions in equation (2.5.4), we get For  $i = 1$  to  $18$ ,  $j = 1$ ,  $k = 0$

$$i = 1, u_{0,1,1} + 4u_{1,1,1} + u_{2,1,1} = 0.2888288$$

$$i = 2, u_{1,1,1} + 4u_{2,1,1} + u_{3,1,1} = 0.5705468$$

$$i = 3, u_{2,1,1} + 4u_{3,1,1} + u_{4,1,1} = 0.8382275$$

$$i = 4, u_{3,1,1} + 4u_{4,1,1} + u_{5,1,1} = 1.0852630$$

$$i = 5, u_{4,1,1} + 4u_{5,1,1} + u_{6,1,1} = 1.3055732$$

$$i = 6, u_{5,1,1} + 4u_{6,1,1} + u_{7,1,1} = 1.4937362$$

$$i = 7, u_{6,1,1} + 4u_{7,1,1} + u_{8,1,1} = 1.6451184$$

$$i = 8, u_{7,1,1} + 4u_{8,1,1} + u_{9,1,1} = 1.7556148$$

$$i = 9, u_{8,1,1} + 4u_{9,1,1} + u_{10,1,1} = 1.8235088$$

$$i = 10, u_{9,1,1} + 4u_{10,1,1} + u_{11,1,1} = 1.8463596$$

$$i = 11, u_{10,1,1} + 4u_{11,1,1} + u_{12,1,1} = 1.8235088$$

$$i = 12, u_{11,1,1} + 4u_{12,1,1} + u_{13,1,1} = 1.7556148$$

$$i = 13, u_{12,1,1} + 4u_{13,1,1} + u_{14,1,1} = 1.6451184$$

$$i = 14, u_{13,1,1} + 4u_{14,1,1} + u_{15,1,1} = 1.4937362$$

$$i = 15, u_{14,1,1} + 4u_{15,1,1} + u_{16,1,1} = 1.3055732$$

$$i = 16, u_{15,1,1} + 4u_{16,1,1} + u_{17,1,1} = 1.0852630$$

$$i = 17, u_{16,1,1} + 4u_{17,1,1} + u_{18,1,1} = 0.8382275$$

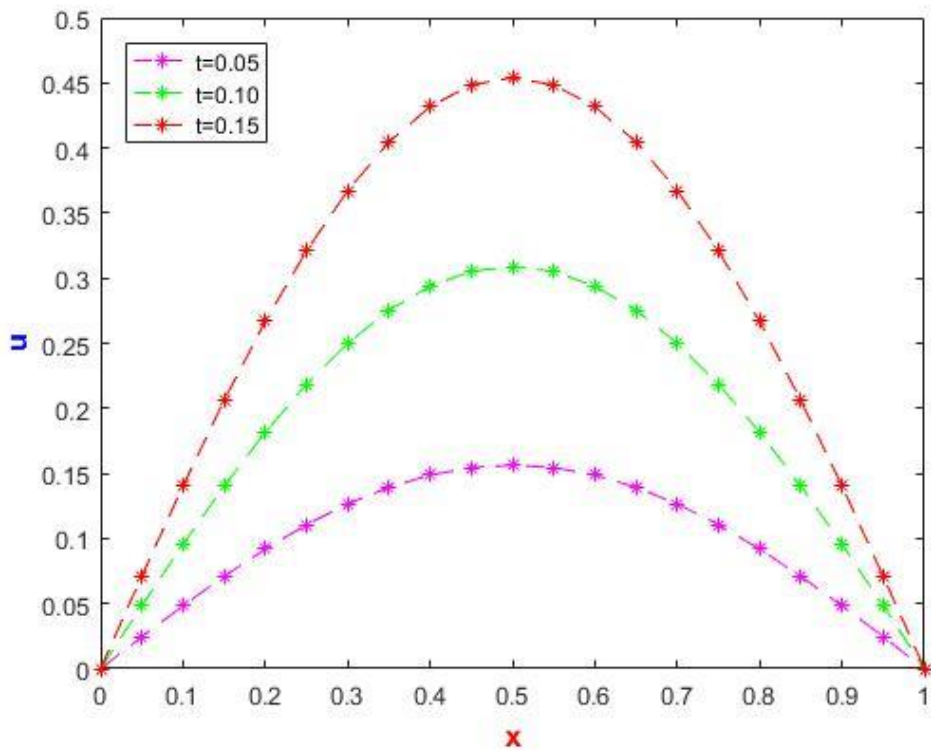
$$i = 18, u_{17,1,1} + 4u_{18,1,1} + u_{19,1,1} = 0.5705468$$

$$i = 19, u_{18,1,1} + 4u_{19,1,1} + u_{20,1,1} = 0.2888288$$

We get  $19 \times 19$  simultaneous equations in  $19 \times 19$  unknowns, where  $i = 1, 2, 3 \dots 19$ . This can be solved by any standard method. Once the result are obtained, for results for  $(k+1)^{\text{th}}$  level, results for  $(k+2)^{\text{th}}$  level are obtained in similar manner discussed as above. Due to the symmetry of the solutions.

**Table: 2.4.1:****Temperature distribution in thin rectangular plate through explicit scheme**

<b>Temperature in thin rectangular plate <math>u(x,t)</math></b>			
<b>x</b>	<b>t = 0.05</b>	<b>t = 0.10</b>	<b>t = 0.15</b>
<b>0</b>	0	0	0
<b>0.05</b>	0.024457632236624	0.048339465678012	0.0710166524123
<b>0.10</b>	0.048329471053505	0.095487124566452	0.1402847542100
<b>0.15</b>	0.070964483549356	0.140284001254612	0.2060978445722
<b>0.20</b>	0.091912594749072	0.181627548648825	0.2668364456877
<b>0.25</b>	0.110585137454357	0.218497001542488	0.3210042587645
<b>0.30</b>	0.126516855433501	0.249988456887821	0.3672689784512
<b>0.35</b>	0.139347440811639	0.275323654869456	0.4044899787456
<b>0.40</b>	0.148723381319943	0.293878456715284	0.4317498567124
<b>0.45</b>	0.154459033908588	0.305197877877210	0.4483798546243
<b>0.50</b>	0.156370483045706	0.309002545542314	0.4539684578301
<b>0.55</b>	0.154459033908588	0.305197877877210	0.4483798546243
<b>0.60</b>	0.148723381319943	0.293878456715284	0.4317498567124
<b>0.65</b>	0.139347440811639	0.275323654869456	0.4044899787456
<b>0.70</b>	0.126516855433501	0.249988456887821	0.3672689784512
<b>0.75</b>	0.110585137454357	0.218497001542488	0.3210042587645
<b>0.80</b>	0.091912594749072	0.181627548648825	0.2668364456877
<b>0.85</b>	0.070964483549356	0.140284001254612	0.2060978445722
<b>0.90</b>	0.048329471053505	0.095487124566452	0.1402847542100
<b>0.95</b>	0.024457632236624	0.048339465678012	0.0710166524123
<b>1</b>	0	0	0



**Figure: 2.4.1:**

**Temperature distribution in thin rectangular plate through explicit scheme**

## 2.4.2 SOLUTIONS WITH SPLINE IMPLICIT SCHEME:

Using Implicit formula given by equation (2.3.7), the solution of equation (2.4.4) satisfying boundary and initial conditions, those are given in section (2.4) is obtained as follows:

Let  $h = 1/20$ ,  $c^2 = 1$  and  $\Delta t = 1/400$  which gives  $r = 0.001$

Hence  $1 + 6r = 1 + 6(0.001) = 1.006$ ,  $4 - 12r = 4 - 12(0.001) = 3.988$

$1 - 6r = 1 - 6(0.001) = 0.994$ ,  $4 + 12r = 4 + 12(0.001) = 4.012$

Substitute the values of  $1+6r$ ,  $1-6r$ ,  $4+12r$  and  $4 - 12r$  with initial and boundary conditions in equation (2.3.7). For  $i = 1, j = 1, k = 0$

$$i = 1, (0.994)u_{0,1,1} + (4.012)u_{1,1,1} + (0.994)u_{2,1,1} = 0.288835$$

$$i = 2, (0.994)u_{1,1,1} + (4.012)u_{2,1,1} + (0.994)u_{3,1,1} = 0.570560$$

$$i = 3, (0.994)u_{2,1,1} + (4.012)u_{3,1,1} + (0.994)u_{4,1,1} = 0.838248$$

$$i = 4, (0.994)u_{3,1,1} + (4.012)u_{4,1,1} + (0.994)u_{5,1,1} = 1.085219$$

$$i = 5, (0.994)u_{4,1,1} + (4.012)u_{5,1,1} + (0.994)u_{6,1,1} = 1.305588$$

$$i = 6, (0.994)u_{5,1,1} + (4.012)u_{6,1,1} + (0.994)u_{7,1,1} = 1.493743$$

$$i = 7, (0.994)u_{6,1,1} + (4.012)u_{7,1,1} + (0.994)u_{8,1,1} = 1.644949$$

$$i = 8, (0.994)u_{7,1,1} + (4.012)u_{8,1,1} + (0.994)u_{9,1,1} = 1.755657$$

$$i = 9, (0.994)u_{8,1,1} + (4.012)u_{9,1,1} + (0.994)u_{10,1,1} = 1.823564$$

$$i = 10, (0.994)u_{9,1,1} + (4.012)u_{10,1,1} + (0.994)u_{11,1,1} = 1.846394$$

$$i = 11, (0.994)u_{10,1,1} + (4.012)u_{11,1,1} + (0.994)u_{12,1,1} = 1.823564$$

$$i = 12, (0.994)u_{11,1,1} + (4.012)u_{12,1,1} + (0.994)u_{13,1,1} = 1.755657$$

$$i = 13, (0.994)u_{12,1,1} + (4.012)u_{13,1,1} + (0.994)u_{14,1,1} = 1.644949$$

$$i = 14, (0.994)u_{13,1,1} + (4.012)u_{14,1,1} + (0.994)u_{15,1,1} = 1.493743$$

$$i = 15, (0.994)u_{14,1,1} + (4.012)u_{15,1,1} + (0.994)u_{16,1,1} = 1.305588$$

$$i = 16, (0.994)u_{15,1,1} + (4.012)u_{16,1,1} + (0.994)u_{17,1,1} = 1.085219$$

$$i = 17, (0.994)u_{16,1,1} + (4.012)u_{17,1,1} + (0.994)u_{18,1,1} = 0.838248$$

$$i = 18, (0.994)u_{17,1,1} + (4.012)u_{18,1,1} + (0.994)u_{19,1,1} = 0.570560$$

$$i = 19, (0.994)u_{18,1,1} + (4.012)u_{19,1,1} + (0.994)u_{20,1,1} = 0.288835$$

We get  $19 \times 19$  simultaneous equations in  $19 \times 19$  unknowns, where  $i = 1, 2, 3, \dots, 19$ . By solving it, the temperature distribution over the plate is obtained. Once the results are obtained, for results for  $(k+1)^{\text{th}}$  level, for  $(k+2)^{\text{th}}$  level, ... are obtained in similar manner discussed as above. Due to the symmetry of the solutions, the results are given for  $0 \leq x \leq 0.5, 0 \leq y \leq 0.5$  at  $1/400, 3/400$  &  $1/80$  respectively in the table.

**Table: 2.4.2:****Temperature distribution in thin rectangular plate through implicit scheme**

<b>Temperature in thin rectangular plate <math>u(x,t)</math></b>			
<b>x</b>	<b>0.05</b>	<b>0.10</b>	<b>0.15</b>
<b>0</b>	0	0	0
<b>0.05</b>	0.024565632236622	0.049039465652412	0.0710166524123
<b>0.10</b>	0.049329471053504	0.095468712456645	0.1402847542100
<b>0.15</b>	0.071024483549351	0.140256001254611	0.2060978445722
<b>0.20</b>	0.091927500149072	0.181627448648825	0.2668364456877
<b>0.25</b>	0.110688137454357	0.218497011542488	0.3210042587645
<b>0.30</b>	0.126527855433501	0.249984456887821	0.3672689784512
<b>0.35</b>	0.139507440811639	0.275335654869456	0.4044899787456
<b>0.40</b>	0.148751381319943	0.293887456715284	0.4317498567124
<b>0.45</b>	0.154470033908588	0.305197877877210	0.4483798546243
<b>0.50</b>	0.156395483045706	0.309002545542314	0.4539684578301
<b>0.55</b>	0.154470033908588	0.305197877877210	0.4483798546243
<b>0.60</b>	0.148751381319943	0.293878456715284	0.4317498567124
<b>0.65</b>	0.139507440811639	0.275323654869456	0.4044899787456
<b>0.70</b>	0.126527855433501	0.249988456887821	0.3672689784512
<b>0.75</b>	0.110588137454357	0.218497001542488	0.3210042587645
<b>0.80</b>	0.091927500149072	0.181627548648825	0.2668364456877
<b>0.85</b>	0.071024483549356	0.140256001254611	0.2060978445722
<b>0.90</b>	0.048329471053505	0.095468712456645	0.1402847542100
<b>0.95</b>	0.024565632236624	0.049039465652412	0.0710166524123
<b>1.00</b>	0	0	0

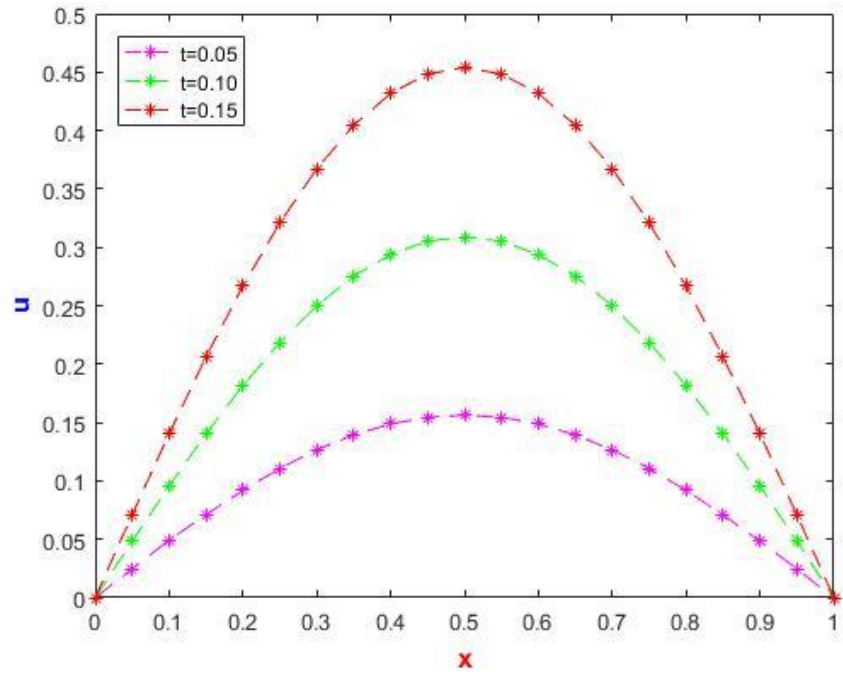
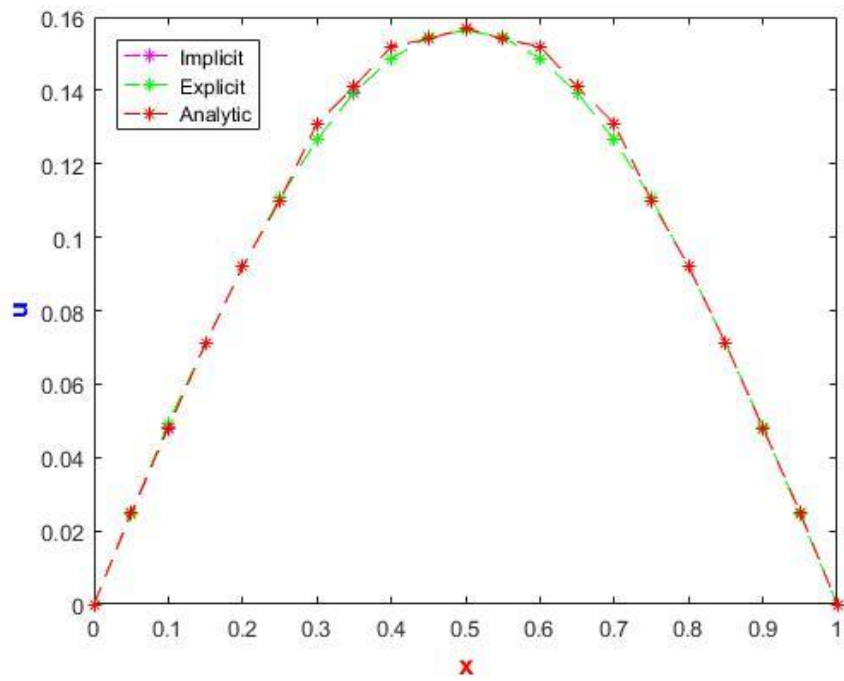


Figure: 2.4.2:

Temperature distribution in thin rectangular plate through implicit scheme

**Table: 2.4.3:****Comparison of Spline explicit and implicit solution with Analytic solution**

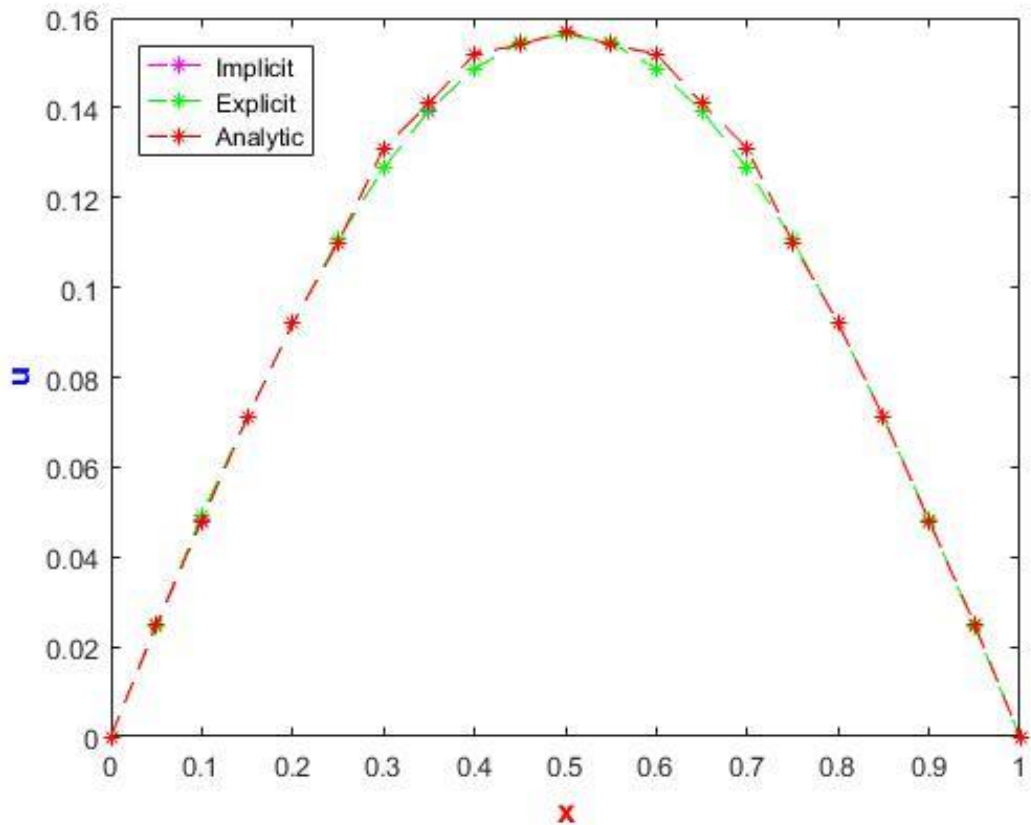
<b>Temperature in thin rectangular plate <math>u(x,t)</math></b>			
<b>x</b>	<b>Explicit Solution</b>	<b>Implicit Solution</b>	<b>Analytic Solution</b>
<b>0</b>	0	0	0
<b>0.05</b>	0.024457632236624	0.024565632236622	0.02458974
<b>0.10</b>	0.048329471053505	0.049329471053504	0.04975410
<b>0.15</b>	0.070964483549356	0.071024483549351	0.07109912
<b>0.20</b>	0.091912594749072	0.091927500149072	0.09192541
<b>0.25</b>	0.110585137454357	0.110688137454357	0.11064442
<b>0.30</b>	0.126516855433501	0.126527855433501	0.12668477
<b>0.35</b>	0.139347440811639	0.139507440811639	0.13951549
<b>0.40</b>	0.148723381319943	0.148751381319943	0.14857451
<b>0.45</b>	0.154459033908588	0.154470033908588	0.15459105
<b>0.50</b>	0.156370483045706	0.156395483045706	0.15658421
<b>0.55</b>	0.154459033908588	0.154470033908588	0.15459105
<b>0.60</b>	0.148723381319943	0.148751381319943	0.14857451
<b>0.65</b>	0.139347440811639	0.139507440811639	0.13951549
<b>0.70</b>	0.126516855433501	0.126527855433501	0.12668477
<b>0.75</b>	0.110585137454357	0.110688137454357	0.11064442
<b>0.80</b>	0.091912594749072	0.091927500149072	0.09192541
<b>0.85</b>	0.070964483549356	0.071024483549356	0.07109912
<b>0.90</b>	0.048329471053505	0.049329471053504	0.04975410
<b>0.95</b>	0.024457632236624	0.024565632236624	0.02458974
<b>1.00</b>	0	0	0



**Figure: 2.4.3:**  
**Comparison of Spline explicit and implicit solution with Analytic solution**

**Table: 2.4.4****Comparison of Spline explicit and implicit solution with Analytic solution**

<b>Temperature in thin rectangular plate <math>u(x,t)</math></b>			
<b>x</b>	<b>Explicit Solution</b>	<b>Implicit Solution</b>	<b>Analytic Solution</b>
<b>0</b>	0	0	0
<b>0.05</b>	0.024433060777528	0.024568784756108	0.02458974
<b>0.10</b>	0.048467756889888	0.049319139635784	0.04975410
<b>0.15</b>	0.070395911662921	0.071025982671758	0.07109912
<b>0.20</b>	0.092048596458428	0.091922219464326	0.09192541
<b>0.25</b>	0.110809702503367	0.110682088589161	0.11064442
<b>0.30</b>	0.126512593528103	0.126511155248281	0.12668477
<b>0.35</b>	0.139339923384220	0.139340302103283	0.13951549
<b>0.40</b>	0.148727712935018	0.148716188909071	0.14857451
<b>0.45</b>	0.154449224875709	0.154451255612298	0.15459105
<b>0.50</b>	0.156375387562145	0.156362902904563	0.15658421
<b>0.55</b>	0.154449224875709	0.154451255612298	0.15459105
<b>0.60</b>	0.148727712935018	0.148716188909071	0.14857451
<b>0.65</b>	0.139339923384220	0.139340302103283	0.13951549
<b>0.70</b>	0.126512593528103	0.126511155248281	0.12668477
<b>0.75</b>	0.110809702503367	0.110682088589161	0.11064442
<b>0.80</b>	0.092048596458428	0.091922219464326	0.09192541
<b>0.85</b>	0.070395911662921	0.071025982671758	0.07109912
<b>0.90</b>	0.048467756889888	0.049319139635784	0.04975410
<b>0.95</b>	0.024433060777528	0.024568784756108	0.02458974
<b>1.00</b>	0	0	0



**Figure: 2.4.4:**

**Comparison of Spline explicit and implicit solution with Analytic solution**

**2.4.3 DISCUSSION OF RESULTS:**

Tables (2.4.3) & (2.4.4) give the comparison of both the solutions obtained by spline explicit and implicit schemes with the available exact solutions. Clearly, the results are accurate up to the five decimal places. This shows reliability of the method.

## 2.5 CONCLUSION:

The results are presented in the tabular form and in graphical manner and the results are compared with the above mentioned two methods (Spline explicit and implicit scheme). The comparison shows that the later method gives quite satisfactory results. We realize that by reducing the value of the size of subinterval gives a better accuracy in the solution. These leads to the conclusion that a better accuracy can be achieved by reducing the mesh size upto a feasible extent.

A linear boundary value problem of order one and two occurring in the study of a thin long rod surrounded except at the ends with a material impervious to heat unless all the points of the rod are at the same temperature and the flow of heat in a thin rectangular plate with sides of length  $\Delta x$  and  $\Delta y$  along co-ordinals  $x$  and  $y$  has been studied. This partial differential equation is treated numerically with iterative spline collocation method. The results presented in table (2.2.3) and (2.4.3) are good for  $j = 0$ . Also the progress of iterative scheme is shown in figure (2.2.4) and (2.4.4) for  $j = 1$ . So it is established here that higher order splines yield fair results.

**CHAPTER -3**  
**APPLICATION OF SPLINE COLLOCATION METHOD TO**  
**HYPERBOLIC EQUATION OF ONE SPACE VARIABLE**  
**AS WELL AS TWO SPACE VARIABLES**

- 3.1 Spline Formula To Solve Hyperbolic Equation with one Space Variables.
- 3.2 Vibrating String Problem.
  - 3.2.1 Solution with Spline Explicit Scheme.
  - 3.2.2 Solution with Spline Implicit Scheme.
  - 3.2.3 Discussion of Result.
- 3.3 Spline Formula To Solve Hyperbolic Equation with Two Space Variables.
- 3.4 Vibrating Membrane Problem.
  - 3.4.1 Solution with Spline Explicit Scheme.
  - 3.4.2 Solution with Spline Implicit Scheme.
  - 3.4.3 Discussion of Result.
- 3.5 Conclusion.

In the previous chapter, we described spline collocation method to solving parabolic partial differential equation of one space and two space variables. This Chapter covers a detailed discussion and solution of one space and two space hyperbolic equations using Spline Explicit and Implicit Scheme.

### 3.1 SPLINE FORMULA TO SOLVE HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS:

The general form of hyperbolic PDE with one space variable  $x$  and time variable  $t$  is given by

$$u_{tt} = c^2 u_{xx} \quad 0 \leq x \leq L, t \geq 0 \quad (3.1.1)$$

with Dirichlet boundary conditions

$$\left. \begin{aligned} u(0, t) &= 0 \\ u(L, t) &= 0 \end{aligned} \right\} \quad (3.1.2)$$

Along with initial conditions

$$\left. \begin{aligned} u(x, 0) &= f(x) \\ u_t(x, 0) &= g(x) \end{aligned} \right\} \quad (3.1.3)$$

In equation (3.1.1),  $c^2$  is a constant term; it depends upon some physical quantities in case of different problem. Divide the region  $0 \leq x \leq L$  into, say  $n$ -equal subinterval each of width  $h$ . The subscript  $j$  denotes time and  $i$  for the positions. Let us denote the points of subdivisions by  $X_0, X_1, X_2, \dots, X_n$ . Let us have the solution on hand at time  $j\Delta t$  at the mesh points  $X_0, X_1, X_2, \dots, X_n$ . Let  $u_{i,j}$  denote the value of  $u$  at the  $(i, j)^{\text{th}}$  mesh point. Discretizing the left hand side of PDE (3.1.1) by the central difference formula and right side by second derivative of cubic spline  $S(x)$ .

$$\frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{\Delta t} = (c^2) S''_{i,j} \quad (3.1.4)$$

where  $S''_{i,j}$  denotes  $S''(x_i)$  at  $(i, j)^{\text{th}}$  level.

Substitute values of  $S''_{i,j}$  from (3.1.4) into equation (1.6.13) and get

$$\begin{aligned} u_{i-1,j+1} + 4u_{i,j+1} + u_{i+1,j+1} &= (2 + 6r^2)u_{i-1,j} + (8 - 12r^2)u_{i,j} + (2 + 6r^2)u_{i+1,j} \\ &\quad - [u_{i-1,j-1} + 4u_{i,j-1} + u_{i+1,j-1}] \end{aligned} \quad (3.1.5)$$

Where  $r^2 = c^2(\Delta t)^2/h^2$   $i = 1(1) n-1$

Above formula is known as **cubic spline explicit formula** to solve hyperbolic PDE of the form (3.1.1). It is clear that above formula is applied for all values of  $j \geq 1$ .

However, for  $j = 0$ , it becomes

$$\begin{aligned} \mathbf{u}_{i-1,1} + 4\mathbf{u}_{i,1} + \mathbf{u}_{i+1,1} = (2 + 6r^2)\mathbf{u}_{i-1,0} + (8 - 12r^2)\mathbf{u}_{i,0} + (2 + 6r^2)\mathbf{u}_{i+1,0} \\ - [\mathbf{u}_{i-1,-1} + 4\mathbf{u}_{i,-1} + \mathbf{u}_{i+1,-1}] \end{aligned} \quad (3.1.6)$$

It involves the terms  $\mathbf{u}_{i+1,-1}$ ,  $\mathbf{u}_{i,-1}$  and  $\mathbf{u}_{i-1,-1}$  which are unknowns. To deal with them, we consider the function  $u = u(x, t)$  to be extended backward in time, the term  $t = t_{-1}$  make a good sense. Most of the time, we get periodic functions for  $u$  versus  $x$  at a given point, we can consider zero time as an arbitrary point at which we know the value of  $u$ . So, to get the values for the fictitious points at  $t = t_{-1}$  we use the initial condition  $u_t = g(x)$  at  $t = 0$

By central difference approximation, we have

$$\mathbf{u}_t(x, 0) = (\mathbf{u}_{i,1} - \mathbf{u}_{i,-1}) / 2\Delta t = g(x_i)$$

Giving  $\mathbf{u}_{i,-1} = \mathbf{u}_{i,1} - 2g(x_i)\Delta t$  at  $t = 0$  only where  $i = 0(1) n-1$

The values of  $\mathbf{u}_{i,-1}$ ;  $i = 0(1)n$  and using initial and boundary conditions with equation (3.1.6) gives system of  $(n-1)$  simultaneous linear equations in  $(n-1)$  unknowns. The system has tri-diagonal matrix, which can be solved by any well-known method. After calculating the values of  $u$  for  $j = 0$ , we apply equation (3.1.5) for  $j \geq 1$ , we get  $(n-1)$  simultaneous linear equations in  $(n-1)$  unknowns with tri-diagonal matrix, again  $r \leq 1$  is the required condition for convergence and stability of this cubic spline explicit method.

Now, we derive implicit scheme for hyperbolic differential equations. Implicit scheme is unconditionally stable, i.e. stable for all the values for  $r$ . In Implicit method the discretization of the differential equation at any mesh point  $(i, j)$  is done by replacing time derivative by the central difference formula and the space derivative is replaced by average of second derivatives of cubic spline  $S(x)$  at the  $(j-1)^{\text{th}}$  and  $(j+1)^{\text{th}}$  level.

$$\frac{\mathbf{u}_{i,j-1} - 2\mathbf{u}_{i,j} + \mathbf{u}_{i,j+1}}{(\Delta t)^2} = \left( \frac{c^2}{2} \right) (S''_{i,j-1} + S''_{i,j+1}) \quad (3.1.7)$$

The values of  $S''_{i,j+1}$  obtained from equation (3.1.7) and with the help of equation (1.6.13) the following relation is obtained.

$$(1-3r^2)u_{i+1,j+1} + (4+6r^2)u_{i,j+1} + (1-3r^2)u_{i-1,j+1} = (3r^2-1)u_{i+1,j-1} - (6r^2+4)u_{i,j-1} + (3r^2-1)u_{i-1,j-1} + 2(u_{i+1,j} + 4u_{i,j} + u_{i-1,j}) \quad (3.1.8)$$

where  $r^2 = c^2(\Delta t)^2/h^2$   $i = 1(1) n-1$

The above equation (3.1.8) is known as **cubic spline implicit formula** to solve hyperbolic PDE of the form (3.1.1). This scheme gives (n-1) unknowns with the coefficient matrix of tri-diagonal form.

For  $j = 0$ , here we can also use the initial condition in similar manner described as above.

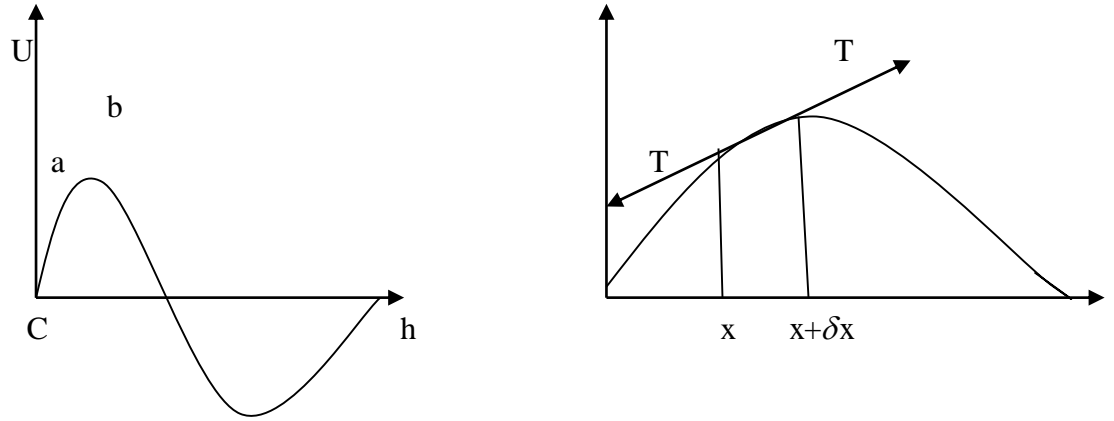
### 3.2 VIBRATING STRING PROBLEM:

Many problems in mathematical physics turn in to the solving of partial differential equations. The partial differential equations play an important role in the theory of elasticity, hydraulics and other some branches of engineering. One of the most important problems in mathematical physics is the vibration of a stretched string. Simplicity and frequent occurrence in many branches of mathematical physics make it classical example in the theory of partial differential equations.

Let us consider a stretched string of length L fixed at the end points. The problem here is to determine the equation of motion which characterizes the position  $u(x, t)$  of the string at time t after an initial disturbance is given.

In order to obtain a simple equation we make the following assumptions.

- (1) The string is flexible and elastic that is the string cannot resist bending moments and thus the tension in the string is always in the direction of the tangent to the existing profile of the string.
- (2) There is no elongation of a single segment of the string and hence by Hook's law the tension is constant.
- (3) The weight of the string is small compared with the tension in the string.
- (4) The deflection is small compared with the length of the string.
- (5) The slope of the displaced string at any point is small compared with unity.
- (6) There is only pure transverse vibration.



**Figure: 3.2.1: String position and tension**

Let  $T$  be the tension at the end points as shown in figure (3.2.1). Consider the vibration of an elastic string of length  $L$  in the vertical plane, the ends of which are fixed. Take the origin at one fixed end, the  $x$ -axis along the length of the string and the  $y$ -axis perpendicular to it. The displacement  $u$  of any point of the string is a function of two variables,  $x$  its distance from  $o$ , and the time  $t$ . To obtain the relation between  $u$ ,  $x$  and  $t$  take a small element  $\delta x$  of the string at a distance  $x$  from the origin of which the displacement by  $y$  at time  $t$ . Let  $m$  be the mass per unit length of the string. Then considering vertical displacement of these elements, its acceleration is  $u_{tt}$ .

If  $\phi$  and  $\phi + \delta\phi$  be the angle made by the tangents at the extremities of this elements, with the  $x$ -axis then the vertical component of the force to which this element is subjected is

$$T \sin(\phi + \delta\phi) - T \sin\phi$$

And as  $\phi$  is small, we can write  $\tan\phi = \sin\phi$ , so these becomes

$$T \tan(\phi + \delta\phi) - T \tan\phi$$

$$T \left\{ \left[ \frac{\delta u}{\delta x} \right]_{x+\delta x} - \left[ \frac{\delta u}{\delta x} \right]_x \right\} \text{ or } \frac{T \left\{ \left[ \frac{\delta u}{\delta x} \right]_{x+\delta x} - \left[ \frac{\delta u}{\delta x} \right]_x \right\}}{\delta x} \delta x \quad (3.2.1)$$

Thus for the vertical motion of this element, we have the equation of motion as

$$(m \cdot \delta x) \frac{\delta^2 u}{\delta t^2} = \frac{T \left\{ \left[ \frac{\delta u}{\delta x} \right]_{x+\delta x} - \left[ \frac{\delta u}{\delta x} \right]_x \right\}}{\delta x} \delta x \quad (3.2.2)$$

Canceling  $\delta x$  on both sides and taking the limit of the equation (3.1.10), we have

$$m \frac{\delta^2 u}{\delta t^2} = T \frac{\delta^2 u}{\delta x^2}$$

$$\frac{\delta^2 u}{\delta t^2} = \frac{T}{m} \frac{\delta^2 u}{\delta x^2} \quad (3.2.3)$$

where  $c^2 = T/m$ ,  $T$  = Tension in a string,  $m$  = Mass per unit length

This is the PDE having the vibrations of the string and is called the WAVE equations, which is hyperbolic type.

Suppose the string of length 2 meter fixed at its both ends, the initial velocity  $g(x)$  is to be taken 0 and from the based on mathematical physics it can be known that the initial displacement  $f(x)$  is  $\sin(\pi x / L)$ . We also assume that constant  $C$  takes value 1, and then this problem becomes an initial boundary value problem defined as

$$u_{tt} = u_{xx} ; 0 \leq x \leq 2 ; t \geq 0 \quad (3.2.4)$$

$$u(x, 0) = \sin\left(\frac{\pi x}{2}\right) \quad 0 \leq x \leq 2 \quad (3.2.5)$$

$$\left. \begin{array}{l} u_t(x, 0) = 0 ; 0 \leq x \leq 2 \\ u(0, t) = 0 ; t \geq 0 \\ u(2, t) = 0 ; t \geq 0 \end{array} \right\} \quad (3.2.6)$$

### 3.2.1 SOLUTIONS WITH SPLINE EXPLICIT METHOD:

Let the string of length 2 meter fixed at both its ends and divide the region with  $h = 0.2$  and let  $\Delta t = (1/20) = 0.05$ ,  $c = 1$  then  $r = 0.25$

$$2 + 6r^2 = 2.375 \text{ And } 8 - 12r^2 = 7.25$$

Substitute the values of  $(2 + 6r^2)$  and  $(8 - 12r^2)$  in equation (3.1.6) and using initial conditions, we get for  $j = 0$ ,  $i = 1$

$$i = 1, u_{0,1} + 4u_{1,1} + u_{2,1} = \frac{1}{2} [(2.375)u_{0,0} + (7.25)u_{1,0} + (2.375)u_{2,0}] = 1.818177$$

$$\text{but } u_{0,1} = 0$$

$$4u_{1,1} + u_{2,1} = 1.818177$$

$$i = 2, u_{1,1} + 4u_{2,1} + u_{3,1} = 3.458383$$

$$i = 3, u_{2,1} + 4u_{3,1} + u_{4,1} = 4.760060$$

$$i = 4, u_{3,1} + 4u_{4,1} + u_{5,1} = 5.595787$$

$$i = 5, u_{4,1} + 4u_{5,1} + u_{6,1} = 5.883759$$

$$i = 6, u_{5,1} + 4u_{6,1} + u_{7,1} = 5.595787$$

$$i = 7, u_{6,1} + 4u_{7,1} + u_{8,1} = 4.760060$$

$$i = 8, u_{7,1} + 4u_{8,1} + u_{9,1} = 3.458386$$

$$i = 9, u_{8,1} + 4u_{9,1} + u_{10,1} = 1.818181$$

$$\text{but } u_{10,1} = 0$$

$$u_{8,1} + u_{9,1} = 1.818181$$

Hence we get nine algebraic equations in nine unknowns with tri-diagonal matrix. This system of equation is solved by using Mat lab. From table (3.2.1) it is clear that, the spline solutions are fairly agree with exact solutions. Similarly, for  $j = 1$ , we get another nine algebraic equations. This can be solved by above method. Proceeding in this way; the results obtained by explicit method are shown in table (3.2.1).

**Table: 3.2.1:**

**Displacement of vibrating string through cubic spline explicit method**

<b>Displacement <math>u(x, t)</math></b>			
<b>x</b>	<b>t = 0.10</b>	<b>t = 0.05</b>	<b>t = 0.00</b>
<b>0</b>	0	0	0
<b>0.2</b>	0.30805507320442	0.30893760027624	0.3090169940
<b>0.4</b>	0.58595670718232	0.58758259889502	0.5877852052
<b>0.6</b>	0.80650109806629	0.80883690414364	0.8090169940
<b>0.8</b>	0.94809890055248	0.95080958453038	0.9510565106
<b>1.0</b>	0.99689029972375	0.99979855773480	1.0000000000
<b>1.2</b>	0.94809890055248	0.95080958453038	0.9510565106
<b>1.4</b>	0.80650109806629	0.80883690414364	0.8090169940
<b>1.6</b>	0.58595670718232	0.58758259889502	0.5877852052
<b>1.8</b>	0.30805507320442	0.30893760027624	0.3090169940
<b>2.0</b>	0	0	0

### 3.2.2 SOLUTIONS WITH SPLINE IMPLICIT METHOD:

Using the same initial and boundary conditions and the values of other parameters discussed in section (3.1) the solution of equation (3.1.8) is obtained by implicit formula that is given by the equation (3.1.1), derived as follows.

For  $i = 1, j = 0$

Let  $h = 0.2$  and  $\Delta t = (1/20) = 0.05$ ,  $c = 1$  then  $r = 0.25$

$$1 - 3r^2 = 1 - 3(0.25)^2 = 0.8125 \text{ and } 4 + 6r^2 = 4 + 6(0.25)^2 = 4.375$$

$$i = 1, (0.8125)u_{0,1} + (4.375)u_{1,1} + (0.8125)u_{2,1} = 1.823849$$

$$\text{but } u_{0,1} = 0 \quad (4.375)u_{1,1} + (0.8125)u_{2,1} = 1.823849$$

$$i = 2, (0.8125)u_{1,1} + (4.375)u_{2,1} + (0.8125)u_{3,1} = 3.469172$$

$$i = 3, (0.8125)u_{2,1} + (4.375)u_{3,1} + (0.8125)u_{4,1} = 4.774909$$

$$i = 4, (0.8125)u_{3,1} + (4.375)u_{4,1} + (0.8125)u_{5,1} = 5.613242$$

$$i = 5, (0.8125)u_{4,1} + (4.375)u_{5,1} + (0.8125)u_{6,1} = 5.902113$$

$$i = 6, (0.8125)u_{5,1} + (4.375)u_{6,1} + (0.8125)u_{7,1} = 5.613242$$

$$i = 7, (0.8125)u_{6,1} + (4.375)u_{7,1} + (0.8125)u_{8,1} = 4.774909$$

$$i = 8, (0.8125)u_{7,1} + (4.375)u_{8,1} + (0.8125)u_{9,1} = 3.469172$$

$$i = 9, (0.8125)u_{8,1} + (4.375)u_{9,1} + (0.8125)u_{10,1} = 1.823849$$

Hence we get nine algebraic equations in nine unknowns with tri-diagonal matrix. Solving above system of linear equations with the help of Mat lab, we get the solution as shown in below table (3.2.2) and these solutions are compared with the exact solutions. Proceeding in this way; the results obtained by implicit method are shown in table (3.2.2).

**Table: 3.2.2:****Displacement of vibrating string through cubic spline implicit method**

<b>Displacement <math>u(x, t)</math></b>			
<b>x</b>	<b>t = 0.05</b>	<b>t = 0.10</b>	<b>t = 0.00</b>
<b>0</b>	0	0	0
<b>0.2</b>	0.3089660751220	0.3080581524312	0.3090169940
<b>0.4</b>	0.5876896309996	0.5859625638318	0.5877852052
<b>0.6</b>	0.8088861194094	0.8065089653975	0.8090169940
<b>0.8</b>	0.9509023414430	0.9481079301815	0.9510565106
<b>1.0</b>	0.9998383487527	0.9969000259325	1.0000000000
<b>1.2</b>	0.9509023414430	0.9481079301815	0.9510565106
<b>1.4</b>	0.8088861194094	0.8065089653975	0.8090169940
<b>1.6</b>	0.5876896309996	0.5859625638318	0.5877852052
<b>1.8</b>	0.3089660751220	0.3080581524312	0.3090169940
<b>2.0</b>	0	0	0

**Table: 3.2.3:****Comparison of Spline explicit solution and implicit solution with Analytic solution**

<b>Displacement <math>u(x, t)</math></b>			
<b>x</b>	<b>Implicit Solution</b>	<b>Explicit Solution</b>	<b>Analytic Solution</b>
<b>0</b>	0	0	0
<b>0.2</b>	0.3089660751220	0.30893760027624	0.3090169940
<b>0.4</b>	0.5876896309996	0.58758259889502	0.5877852052
<b>0.6</b>	0.8088861194094	0.80883690414364	0.8090169940
<b>0.8</b>	0.9509023414430	0.95080958453038	0.9510565106
<b>1.0</b>	0.9998383487527	0.99979855773480	1.0000000000
<b>1.2</b>	0.9509023414430	0.95080958453038	0.9510565106
<b>1.4</b>	0.8088861194094	0.80883690414364	0.8090169940
<b>1.6</b>	0.5876896309996	0.58758259889502	0.5877852052
<b>1.8</b>	0.3089660751220	0.30893760027624	0.3090169940
<b>2.0</b>	0	0	0

**Table: 3.2.4:**

**Comparison of Spline explicit solution and implicit solution with Analytic solution**

<b>Displacement <math>u(x, t)</math></b>			
<b>x</b>	<b>Implicit Solution</b>	<b>Explicit Solution</b>	<b>Analytic Solution</b>
<b>0</b>	0	0	0
<b>0.2</b>	0.3080383648	0.3049722720	0.3090169940
<b>0.4</b>	0.5858857278	0.5810859116	0.5877852052
<b>0.6</b>	0.8064846392	0.7964340814	0.8090169940
<b>0.8</b>	0.9480585221	0.9484027624	0.9510565106
<b>1.0</b>	0.9968925488	0.9532048687	1.0000000000
<b>1.2</b>	0.9480585221	0.9484027624	0.9510565106
<b>1.4</b>	0.8064846392	0.7964340814	0.8090169940
<b>1.6</b>	0.5858857278	0.5810859116	0.5877852052
<b>1.8</b>	0.3080383648	0.3049722720	0.3090169940
<b>2.0</b>	0	0	0

**3.2.3 DISCUSSION OF RESULT:**

The result presented in table (3.2.1) and (3.2.2) is obtained by explicit and implicit methods are compared with the exact solutions. From the table (3.2.3) and (3.2.4), it is clear that solutions obtained by both the methods are fairly good. Also implicit method gives better results than the explicit one.

### 3.3 SPLINE FORMULA TO SOLVE HYPERBOLIC PARTIAL DIFFERENTIAL EQUATION WITH TWO SPACE VARIABLE:

The general form of hyperbolic differential equation having two space variables is as follows:

$$u_{tt} = c^2(u_{xx} + u_{yy}) \quad (3.3.1)$$

With Dirichlet boundary conditions namely,

$$\left. \begin{aligned} u(0, y, t) &= 0 \\ u(x, 0, t) &= 0 \end{aligned} \right\} \quad (3.3.2)$$

And initial conditions at  $t = 0$  (Cauchy Conditions)

$$\left. \begin{aligned} u(x, y, 0) &= f(x, y) \\ u_t(x, y, 0) &= g(x, y) \end{aligned} \right\} \quad (3.3.3)$$

Where  $c^2$  is a constant term, it depends upon some physical quantities in case of different types of problems. Divide the region  $0 \leq x \leq a$  into say  $n$  sub-intervals each width  $\Delta x$  such that  $n\Delta x = a$ ,  $0 \leq y \leq b$  into  $m$  subintervals each of width  $\Delta y$  such that  $m\Delta y = b$ .

The subscript  $k$  denotes the time and  $(i, j)$  denote the position. For simplicity consider square region i.e.  $a = b = L$  the length of the region,  $m = n$  and  $\Delta x = \Delta y = h$ . The points of subdivisions are  $(x_i, y_j)$ .

For explicit scheme, the formula is obtained in similar manner as discussed in section (3.1), by replacing the right side by twice the second derivative of cubic spline  $S(x)$  at the  $(i, j)^{\text{th}}$  mesh point.

$$u_{i-1,j,k+1} + 4u_{i,j,k+1} + u_{i+1,j,k+1} = (2 + 12r^2)u_{i+1,j,k} + (8 - 24r^2)u_{i,j,k} + (2 + 12r^2)u_{i-1,j,k} - [u_{i+1,j,k-1} + 4u_{i,j,k-1} + u_{i-1,j,k-1}] \quad (3.3.4)$$

where  $r^2 = c^2(\Delta t)^2/h^2$   $i = 1(1) n-1$

The values of  $u_{i,j,-1}$   $i, j = 1(1) n-1$  are obtained from following relation

$$u_{i,j,-1} = u_{i,j,-1} - 2g(x_i, y_j)\Delta t \quad (3.3.5)$$

Equation (3.3.4) is known as **cubic spline explicit formula** to solve hyperbolic type PDE of the form given by equation (3.3.1). The values obtained from equation (3.3.5) using initial and boundary conditions, the equation (3.3.4) gives system of  $(n-1) \times (n-1)$  simultaneous equations in  $(n-1) \times (n-1)$  unknowns. After the calculating results for  $k = 0$ ; the results for  $k = 1, 2 \dots n$  are obtained in similar manner.  $r \leq \frac{1}{2}$  is the required condition for convergence and stability of this explicit method.

We have implicit scheme to solve hyperbolic differential equation with two space variables. This implicit scheme is unconditionally stable. As similar technique discussed in section (3.2) the cubic spline implicit formula to solve equation (3.3.1) is as follows, by replacing the right side of equation (3.3.1).

$$\begin{aligned}
 & (1 - 6r^2)u_{i-1,j,k+1} + (4 + 12r^2)u_{i,j,k+1} + (1 - 6r^2)u_{i+1,j,k+1} \\
 & = (6r^2 + 1)u_{i-1,j,k} + (12r^2 + 4)u_{i,j,k} + (6r^2 + 1)u_{i+1,j,k} + 2[u_{i+1,j,k} + 4u_{i,j,k} + u_{i-1,j,k}]
 \end{aligned} \tag{3.3.6}$$

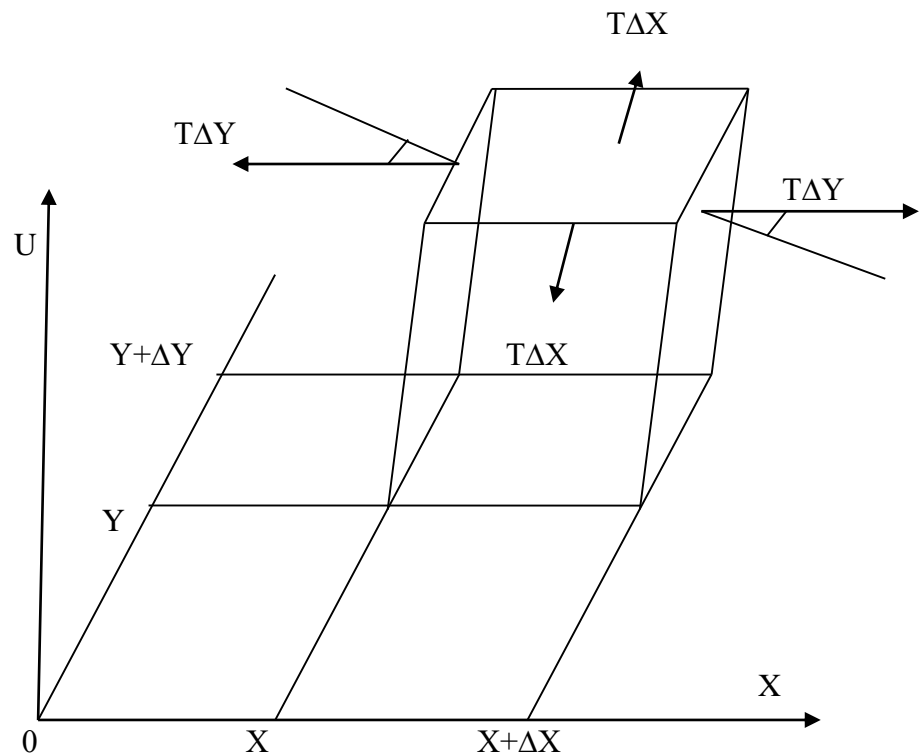
where  $r^2 = c^2(\Delta t)^2/h^2$  ;  $i, j = 1(1) n-1$

Equation (3.3.6) gives  $(n - 1) \times (n - 1)$  simultaneous equations in  $(n - 1) \times (n - 1)$  unknowns. After calculating results for  $(k+1)^{\text{th}}$  time level, the results for  $(k+2)^{\text{th}}$  time level are obtained in similar manner by repeating the process.

### 3.4 VIBRATING MEMBRANE PROBLEM:

The problem of vibrating membrane is well-known to researchers. Assume that membrane is tightly stretched and homogenous i.e. its mass per unit area is constant, it is perfectly flexible and is so thin that it offers no resistance for bending. The membrane is stretched and then fixed along its entire boundary in the  $xy$ -plane and the tension  $T$  caused by stretching the membrane is the same at every point in all directions and does not change during the motion. The deflection  $u(x, y, t)$  of the membrane during the motion is small compared to the size of the membrane.

Consider the forces acting on a small portion of the membrane as shown in below figure Since the deflection of the membrane and the angles of inclination are small the sides of the portion may be taken approximately equal to  $\Delta x$  and  $\Delta y$ .



**Figure: 3.4.1: Vibrating membrane**

The tension  $T$  is the force per unit length. Forces acting on the edges are  $T\Delta X$  and  $T\Delta Y$  which could be taken tangent to the membrane since the membrane is perfectly elastic. Let the force  $T\Delta Y$  make angles  $\alpha$  and  $\beta$  with the horizontal on the opposite edges of the membrane since the membrane is perfectly elastic.

The resultant vertical component of force due to  $T\Delta Y$  is therefore, up to a first order approximation.

$$\begin{aligned}
& (T\Delta Y)\sin\beta - (T\Delta Y)\sin\alpha \\
& (T\Delta Y)\tan\beta - (T\Delta Y)\tan\alpha \\
& = T\Delta Y \{ (\partial u / \partial x)_{x+\Delta x} - (\partial u / \partial x)_x \} \\
& = T\Delta Y (\partial^2 u / \partial x^2) \Delta x
\end{aligned}$$

Note that sine have replaced by tangent because the angles  $\alpha$  and  $\beta$  are small.

Similarly the forces  $T\Delta X$  acting on the edges of length  $\Delta X$  can be shown to have the vertical component.

$$T\Delta x (\partial^2 u / \partial y^2) \Delta y$$

If  $m$  be the mass per unit area of the membrane, by Newton's second law of motion

$$(m\Delta x\Delta y) \partial^2 u / \partial t^2 = T (\partial^2 u / \partial x^2 + \partial^2 u / \partial y^2) \Delta x \Delta y$$

$$\partial^2 u / \partial t^2 = c^2 (\partial^2 u / \partial x^2 + \partial^2 u / \partial y^2) \quad (3.4.1)$$

where  $c^2 = T/m$

The above equation (3.4.1) is hyperbolic PDE in two space variables  $x$  and  $y$ , and time variable  $t$ . The solution for the case of a rectangular membrane is given below by cubic spline explicit as well as implicit method using following initial and boundary conditions.

Dirichlet boundary conditions:

$$\left. \begin{aligned}
& u(0, y, t) = 0 ; 0 \leq y \leq t \\
& u(x, 0, t) = 0 ; 0 \leq x \leq t
\end{aligned} \right\} \quad (3.4.2)$$

Initial conditions at  $t = 0$  (Cauchy conditions)

$$\left. \begin{aligned}
& u(x, y, 0) = f(x, y) = \sin(\pi x) \cdot \sin(\pi y) ; 0 \leq x, y \leq 1 \\
& u(x, y, 0) = 0
\end{aligned} \right\} \quad (3.4.3)$$

### 3.4.1 SOLUTIONS WITH SPLINE EXPLICIT METHOD:

Using initial and boundary conditions are described by equation (3.4.2) and (3.4.3), the solution of equations (3.4.1) is obtained through explicit formula.

Let the region  $0 \leq x \leq 1$  and  $0 \leq y \leq 1$  be divided into ten sub-intervals, each of length  $h = 0.1$ ,  $c = 0.1$  and  $\Delta t = 0.01$ , these gives  $r = 0.01$ .

For  $i = 1$  to  $9$ ,  $j = 1$  and  $k = 0$  then

$$2 + 12r^2 = 2 + 12(0.01)^2 = 2.0012, \quad 8 - 24r^2 = 8 - 24(0.01)^2 = 7.9976$$

$$i = 1, u_{0,1,1} + 4u_{1,1,1} + u_{2,1,1} = 0.563599$$

$$\text{Since } u_{0,1,1} = 0$$

$$4u_{1,1,1} + u_{2,1,1} = 0.563599$$

$$i = 2, u_{1,1,1} + 4u_{2,1,1} + u_{3,1,1} = 1.072029$$

$$i = 3, u_{2,1,1} + 4u_{3,1,1} + u_{4,1,1} = 1.475521$$

$$i = 4, u_{3,1,1} + 4u_{4,1,1} + u_{5,1,1} = 1.734579$$

$$i = 5, u_{4,1,1} + 4u_{5,1,1} + u_{6,1,1} = 1.873844$$

$$\text{Since } u_{4,1,1} = u_{6,1,1} \text{ (Due to symmetry of the problem)}$$

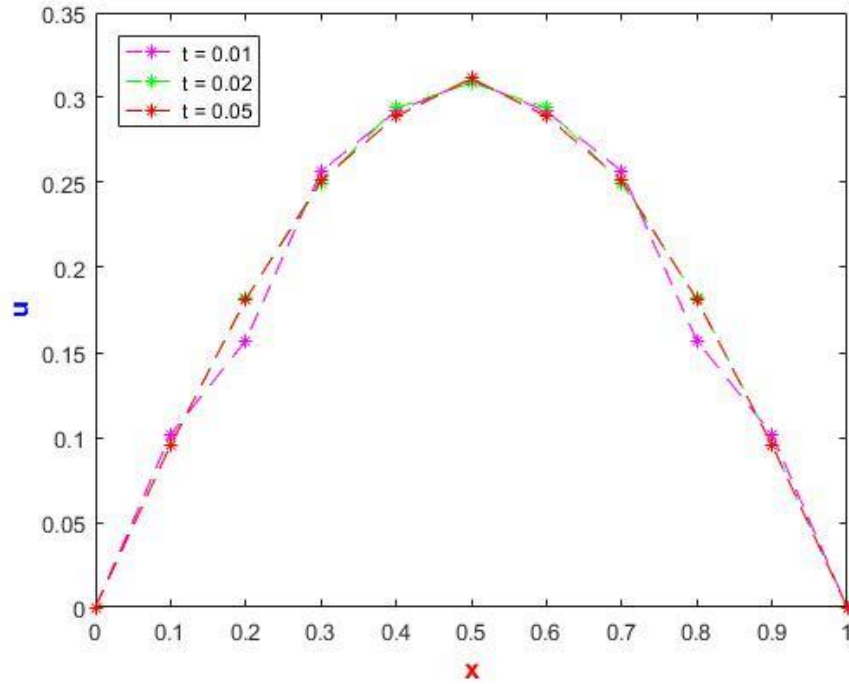
$$2u_{4,1,1} + 4u_{5,1,1} = 1.873844$$

In similar manner calculate the equation  $i = 1(1)9$  one gets  $9 \times 9$  simultaneous linear equations in  $9 \times 9$  unknowns. This system of equations can be solved by any well-known method. Once the values of  $u$  are known at the first level of time, the process can be repeated for second time level and so on. The results obtained by explicit method are given as follows. Due to symmetry of the problem, results are given for  $0 \leq y \leq 0.5, 0 \leq x \leq 0.5$  in the table.

**Table: 3.4.1:**

**Velocity distribution in rectangular vibrating membrane through  
Cubic spline explicit method**

<b>X</b>	<b>t = 0.01</b>	<b>t = 0.02</b>	<b>t = 0.05</b>
<b>0</b>	0	0	0
<b>0.1</b>	0.1016581498	0.0954722145	0.0955509336
<b>0.2</b>	0.1568850805	0.1816386016	0.1812054154
<b>0.3</b>	0.2566293379	0.2497618689	0.2511927694
<b>0.4</b>	0.2919468477	0.2939358907	0.2890196767
<b>0.5</b>	0.3099470311	0.3088066421	0.3113014866
<b>0.6</b>	0.2919468477	0.2939358907	0.2890196767
<b>0.7</b>	0.2566293379	0.2497618689	0.2511927694
<b>0.8</b>	0.1568850805	0.1816386016	0.1812054154
<b>0.9</b>	0.1016581498	0.0954722145	0.0955509336
<b>1.0</b>	0	0	0



**Figure: 3.4.1:**

**Vibrating distribution in rectangular membrane through cubic spline explicit method**

**3.4.2 SOLUTIONS WITH SPLINE IMPLICIT METHOD:**

In similar manner discussed as earlier, using initial and boundary condition, at  $\Delta t = 0.01$ ,  $r = 0.01$ , the solution of equation (3.4.1) is obtained by using cubic spline implicit formula given by equation (3.4.6) as follows.

$$1 - 6r^2 = 1 - 6(0.01)^2 = 0.9994, \quad 4 + 12r^2 = 4 + 12(0.01)^2 = 4.0012$$

$$1 + 6r^2 = 1 + 6(0.01)^2 = 1.0006, \quad 4 - 12r^2 = 4 - 12(0.01)^2 = 3.9988$$

For  $i = 1, j = 1$  and  $k = 0$

$$i = 1, (0.9994)u_{0,1,1} + (4.0012)u_{1,1,1} + (0.9994)u_{2,1,1} = 0.563602$$

$$i = 2, (0.9994)u_{1,1,1} + (4.0012)u_{2,1,1} + (0.9994)u_{3,1,1} = 1.072034$$

$$i = 3, (0.9994)u_{2,1,1} + (4.0012)u_{3,1,1} + (0.9994)u_{4,1,1} = 1.475528$$

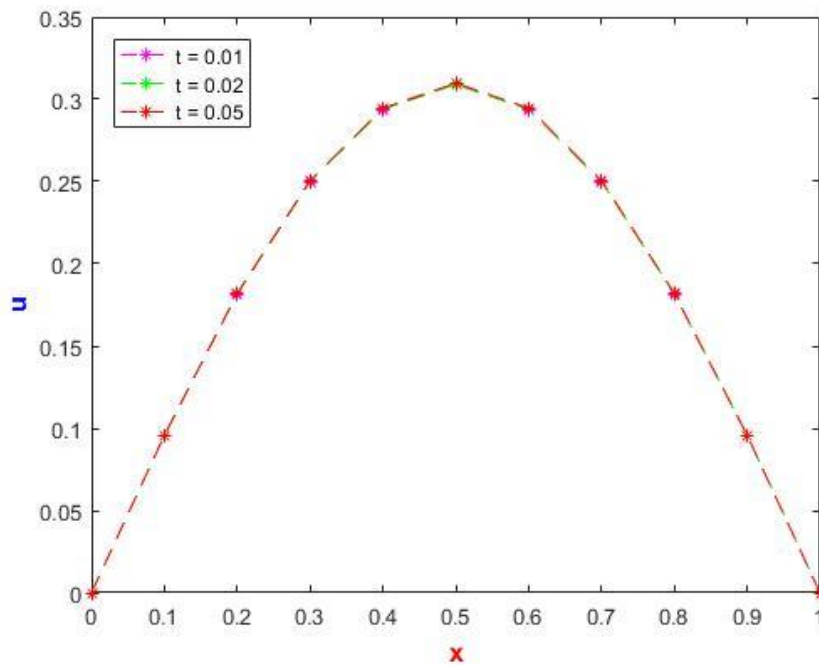
$$i = 4, (0.9994)u_{3,1,1} + (4.0012)u_{4,1,1} + (0.9994)u_{5,1,1} = 1.734587$$

$$i = 5, (1.9988)u_{4,1,1} + (4.0012)u_{5,1,1} = 1.072034$$

Like explicit scheme discussed in above section, one gets system of  $9 \times 9$  simultaneous linear equation in  $9 \times 9$  unknowns. Once the values of  $u$  are obtained at the first time level, the values for second time level are obtained in similar manner by repeating process. Results obtained by cubic spline implicit method are given at different times in the table (3.4.2).

**Table: 3.4.2:**  
**Velocity distribution in rectangular vibrating membrane through**  
**Cubic spline implicit method**

<b>X</b>	<b>t = 0.01</b>	<b>t = 0.02</b>	<b>t = 0.05</b>
<b>0</b>	0	0	0
<b>0.1</b>	0.095490000	0.09551882219	0.0957302328
<b>0.2</b>	0.181590000	0.18164370760	0.1820375685
<b>0.3</b>	0.249981000	0.25005608837	0.2506068430
<b>0.4</b>	0.293850000	0.29393727977	0.2945773591
<b>0.5</b>	0.309000000	0.30909269387	0.3097725704
<b>0.6</b>	0.293850000	0.29393727977	0.2945773591
<b>0.7</b>	0.249981000	0.25005608837	0.2506068430
<b>0.8</b>	0.181590000	0.18164370760	0.1820375685
<b>0.9</b>	0.095490000	0.09551882219	0.0957302328
<b>1.0</b>	0	0	0

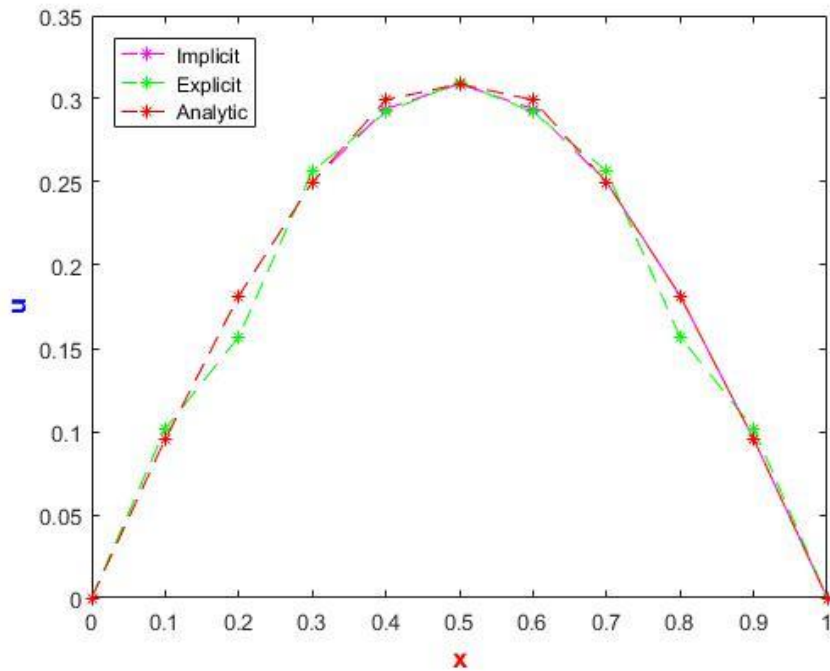


**Figure: 3.4.2:**  
**Vibrating distribution in rectangular membrane through cubic spline implicit**  
**method**

**Table: 3.4.3:**

**Comparison of Spline explicit solution and implicit solution with Analytic solution**

<b>X</b>	<b>Spline Implicit Solution</b>	<b>Spline Explicit Solution</b>	<b>Analytic Solution</b>
<b>0</b>	0	0	0
<b>0.1</b>	0.095490000	0.1016581498	0.095389
<b>0.2</b>	0.181590000	0.1568850805	0.181498
<b>0.3</b>	0.249981000	0.2566293379	0.249874
<b>0.4</b>	0.293850000	0.2919468477	0.299357
<b>0.5</b>	0.309000000	0.3099470311	0.308855
<b>0.6</b>	0.293850000	0.2919468477	0.299357
<b>0.7</b>	0.249981000	0.2566293379	0.249874
<b>0.8</b>	0.181590000	0.1568850805	0.181498
<b>0.9</b>	0.095490000	0.1016581498	0.095389
<b>1.0</b>	0	0	0



**Figure: 3.4.3:**

**Comparison of Spline explicit solution and implicit solution with Analytic solution**

### **3.4.3 DISCUSSION OF RESULT:**

Table (3.4.3) gives the comparison of both the spline solutions namely explicit and implicit with available analytic solutions. From table (3.4.3) it is clear that, the spline solutions are equitably agree with analytic solutions. The figure (3.4.3) gives good agreement of curves presenting exact and approximate solutions obtained by spline method.

### **3.5 CONCLUSION:**

In this chapter the method of spline collocation is applied to linear PDE of hyperbolic type with one space variable and two space variables. The problem describing vibrating string and vibrating membrane are discussed briefly. The equations are solved by explicit and implicit schemes of spline functions. The results are given in tabular form. Comparison of the results shows that the spline solutions are quite reliable. The figure provides a pictorial evidence of a good agreement of the curves presenting actual as well as approximate solutions. Results are accurate up to three digits of decimal place in most of case.

An ultimate conclusion is drawn from this work is that, both the schemes of spline collocation gives accurate results with compact computations.

## **CHAPTER - 4**

### **APPLICATION OF SPLINE COLLOCATION METHOD TO THE BURGERS' EQUATIONS OF ONE, TWO AS WELL AS THREE DIMENSIONALS**

- 4.1 Spline formula to Solve one dimensional Burger's equation
  - 4.1.1 Solution with Spline Explicit Scheme
  - 4.1.2 Solution with Spline Implicit Scheme
  - 4.1.3 Discussion of Result
- 4.2 Spline formula to Solve two dimensional Burger's equation
  - 4.2.1 Solution with Spline Explicit Scheme (When  $\alpha(t) = 0$ )
  - 4.2.2 Solution with Spline Implicit Scheme (When  $\alpha(t) = 0$ )
  - 4.2.3 Solution with Spline Explicit Scheme (When  $\alpha(t) = 1$ )
  - 4.2.4 Solution with Spline Implicit Scheme (When  $\alpha(t) = 1$ )
  - 4.2.5 Discussion of Result
- 4.3 Spline formula to solve Three dimensional Burger's equation
  - 4.3.1 Solution with Spline Explicit Scheme
  - 4.3.2 Solution with Spline Implicit Scheme
  - 4.3.3 Discussion of Result
- 4.4 Conclusion

The Spline Collocation method is developed for semi linear and quasilinear cases, however, it can easily be extended to other type of nonlinearities as well. In this chapter we have to illustrate the method for nonlinear one, two and three dimensional Burgers' equation. The Spline Collocation Method is implemented based of Cole-Hopf transformation. The Cole-Hopf transformation turning the strongly nonlinear Burgers equation into the linear heat equation plays an important role in the development of mathematical sciences. This Chapter covers a detailed discussion and solution of one, two and three dimensional Burgers' equations using Spline Explicit and Implicit Scheme.

#### 4.1 SPLINE FORMULA TO SOLVE ONE DIMENSIONAL VISCOUS BURGERS' EQUATION:

The one dimensional non-linear viscous burgers' equation is

$$\frac{\partial}{\partial t} u(x, t) + u(x, t) \frac{\partial}{\partial x} u(x, t) = \varepsilon \frac{\partial^2}{\partial x^2} u(x, t) \quad (4.1.1)$$

with boundary conditions

$$\left. \begin{array}{l} u(0, t) = 0 \\ u(1, t) = 0 \end{array} \right\} t > 0 \quad (4.1.2)$$

along with initial conditions

$$u(x, 0) = \sin(\pi x) \quad ; 0 \leq x \leq 1 \quad (4.1.3)$$

In equation (4.1.1),  $\varepsilon = (1/Re)$  for simulating the physical phenomena of wave motion and thus determine the behavior of the solutions. Divide the region  $0 \leq x \leq 1$  into, say n-equal subinterval each of width  $h (= \Delta x)$ . The subscript  $j$  denotes time and  $i$  for the positions. Let us denote the points of subdivisions by  $X_0, X_1, X_2, \dots, X_n$ . Let us have the solution on hand at time  $j\Delta t$  at the mesh points  $X_0, X_1, X_2, \dots, X_n$ . Let  $u_{i,j}$  denote the value of  $u$  at the  $(i, j)^{th}$  mesh point.

So first we reduce this nonlinear equation to linear form using Cole-Hopf transformation

$$u(x, t) = - 2\varepsilon \frac{\phi_x}{\phi}$$

So that equation (4.1.1) becomes

$$\frac{\partial \phi}{\partial t} = \varepsilon \frac{\partial^2 \phi}{\partial x^2}, \quad 0 \leq x \leq 1, t > 0 \quad (4.1.4)$$

With the initial condition

$$\phi(x,0) = \exp\{-(2\pi\varepsilon)^{-1}[1-\cos(\pi x)]\}, \quad 0 \leq x \leq 1 \quad (4.1.5)$$

And boundary condition

$$\phi_x(0,t) = \phi_x(1,t) = 0, \quad t > 0 \quad (4.1.6)$$

This means that  $\phi(x, t)$  is any solution of the heat equation (4.1.4) subject to the condition (4.1.5) and (4.1.6), then the transformation is a solution of the Burgers equation (4.1.1) with the condition (4.1.3).

Now discretizing the left hand side of PDE (4.1.4) by forward difference formula like finite difference and right side by second derivative of cubic spline  $S(x)$ .

$$\frac{\phi_{i,j+1} - \phi_{i,j}}{\Delta t} = (\varepsilon) S''_{i,j} \quad (4.1.7)$$

Where  $S''_{i,j}$  denotes  $S''(x_i)$  at  $(i, j)$ <sup>th</sup> level.

Substitute values of  $S''_{i,j}$  in below equation

$$S''_{i-1,j} + 4S''_{i,j} + S''_{i+1,j} = \left(\frac{6}{h^2}\right)(\phi_{i-1,j} - 2\phi_{i,j} + \phi_{i+1,j})$$

Then we get

$$\begin{aligned} \frac{(\phi_{i-1,j+1} - \phi_{i-1,j})}{\varepsilon \Delta t} + 4 \frac{(\phi_{i,j+1} - \phi_{i,j})}{\varepsilon \Delta t} + \frac{(\phi_{i+1,j+1} - \phi_{i+1,j})}{\varepsilon \Delta t} &= \left(\frac{6}{h^2}\right) [\phi_{i-1,j} - 2\phi_{i,j} + \phi_{i+1,j}] \\ \phi_{i-1,j+1} + 4\phi_{i,j+1} + \phi_{i+1,j+1} &= \left(1 + 6 \frac{\varepsilon \Delta t}{h^2}\right) \phi_{i-1,j} + \left(4 - 12 \frac{\varepsilon \Delta t}{h^2}\right) \phi_{i,j} + \left(1 + 6 \frac{\varepsilon \Delta t}{h^2}\right) \phi_{i+1,j} \end{aligned} \quad (4.1.8)$$

Above formula (4.1.8) is known as spline explicit formula at to solve one dimensional Burger's equation of the form (4.1.1).

We have implicit scheme for one dimensional Burger's equations. In Implicit scheme is unconditionally stable In Implicit method the discretization of the differential equation at any mesh point  $(i, j)$  is done by replacing time derivative by the forward difference formula as done in explicit scheme and the space derivative is replaced by average of second derivatives of cubic spline  $S(x)$  at the  $(j)$ <sup>th</sup> and  $(j+1)$ <sup>th</sup> level.

$$\frac{\phi_{i,j+1} - \phi_{i,j}}{\Delta t} = \left(\frac{\varepsilon}{2}\right) (S''_{i,j} + S''_{i,j+1}) \quad (4.1.9)$$

The values of  $S_{i,j+1}''$  obtained from equation (1.6.13) and with the help of below equations the following relation is obtained.

$$\begin{aligned}
S_{i-1,j}'' + 4S_{i,j}'' + S_{i+1,j}'' &= \left(\frac{6}{h^2}\right)(\phi_{i-1,j} - 2\phi_{i,j} + \phi_{i+1,j}) \\
S_{i-1,j+1}'' + 4S_{i,j+1}'' + S_{i+1,j+1}'' &= \left(\frac{6}{h^2}\right)(\phi_{i-1,j+1} - 2\phi_{i,j+1} + \phi_{i+1,j+1}) \\
\left(1-3\frac{\varepsilon\Delta t}{h^2}\right)\phi_{i-1,j+1} + \left(4+6\frac{\varepsilon\Delta t}{h^2}\right)\phi_{i,j+1} + \left(1-3\frac{\varepsilon\Delta t}{h^2}\right)\phi_{i+1,j+1} \\
&= \left(1+3\frac{\varepsilon\Delta t}{h^2}\right)\phi_{i-1,j} + \left(4-6\frac{\varepsilon\Delta t}{h^2}\right)\phi_{i,j} + \left(1+3\frac{\varepsilon\Delta t}{h^2}\right)\phi_{i+1,j}
\end{aligned} \tag{4.1.10}$$

Above formula (4.1.10) is known as spline explicit formula at to solve one dimensional Burger's equation of the form (4.1.1).

#### 4.1.1 SOLUTIONS WITH SPLINE EXPLICIT SCHEME:

Let the  $\varepsilon = 1$  and divide the region with  $h = 0.1$  and  $\Delta t = 0.00001$

$$1 + (6\Delta t/h^2) = 1.006 \quad \& \quad 4 - (12\Delta t/h^2) = 3.988$$

Substitute the values of  $[1 + (6\Delta t/h^2)]$  and  $[4 - (12\Delta t/h^2)]$  in equation (4.1.8) and using initial conditions, we get for  $j = 0, i = 1$

$$\begin{aligned}
i = 1, \phi_{0,1} + 4\phi_{1,1} + \phi_{2,1} &= [(1.006)u_{0,0} + (3.988)u_{1,0} + (1.006)u_{2,0}] = 1.80570 \\
i = 2, \phi_{1,1} + 4\phi_{2,1} + \phi_{3,1} &= 3.434653 \\
i = 3, \phi_{2,1} + 4\phi_{3,1} + \phi_{4,1} &= 4.727394 \\
i = 4, \phi_{3,1} + 4\phi_{4,1} + \phi_{5,1} &= 5.557385 \\
i = 5, \phi_{4,1} + 4\phi_{5,1} + \phi_{6,1} &= 5.843380 \\
i = 6, \phi_{5,1} + 4\phi_{6,1} + \phi_{7,1} &= 5.557385 \\
i = 7, \phi_{6,1} + 4\phi_{7,1} + \phi_{8,1} &= 4.727394 \\
i = 8, \phi_{7,1} + 4\phi_{8,1} + \phi_{9,1} &= 3.434653 \\
i = 9, \phi_{8,1} + 4\phi_{9,1} + \phi_{10,1} &= 1.805703
\end{aligned}$$

Solving above system of nine equations in nine unknowns with the help of Mat lab, we get the solution as shown in table (4.1.1) and plotted in figure. These solutions are compared with the exact solutions. From table (4.1.3) it is clear that, the spline solutions are fairly agree with exact solutions up to ten digits of decimal points, Similarly, for  $j = 1$ , we get another nine algebraic equations. This can be solved by above method.

**Table: 4.1.1:**

**Solution of One Dimensional Burger's equation using Cubic Spline explicit scheme**

<b>u(x,t)</b>			
<b>x</b>	<b>t = 1/1000</b>	<b>t = 2/1000</b>	<b>t = 3/1000</b>
0	0	0	0
0.1	0.305940883977901	0.302818342541436	0.299727606862460
0.2	0.581936464088398	0.576066629834254	0.570249572550160
0.3	0.800966259668508	0.792835138121547	0.784754102936900
0.4	0.941592497237569	0.932132817679558	0.922694015702239
0.5	0.990048751381216	0.980033591160221	0.970069834254144
0.6	0.941592497237569	0.932132817679558	0.922626647281186
0.7	0.800966259668508	0.792835138121547	0.784754102936900
0.8	0.581936464088398	0.576066629834254	0.570249572550160
0.9	0.305940883977901	0.302818342541436	0.299727606862460
1.0	0	0	0

#### 4.1.2 SPLINE SOLUTIONS WITH IMPLICIT METHOD:

Using the same initial and boundary conditions the solution of equation (4.1.1) is obtained by implicit formula that is given by the equation (4.1.8), derived as follows.

For  $i = 1, j = 0$

Let  $h = 0.1$  and  $\Delta t = (1/1000) = 0.001$ ,  $\varepsilon = 1$  then

$$1 - (3\Delta t/h^2) = 0.7 \text{ and } 4 + (6\Delta t/h^2) = 4.6$$

$$1 + (3\Delta t/h^2) = 1.3 \text{ and } 4 - (6\Delta t/h^2) = 3.4$$

$$i = 1, (0.7)\phi_{0,1} + (4.6)\phi_{1,1} + (0.7)\phi_{2,1} = 1.81461$$

$$i = 2, (0.7)\phi_{1,1} + (4.6)\phi_{2,1} + (0.7)\phi_{3,1} = 3.45158$$

$$i = 3, (0.7)\phi_{2,1} + (4.6)\phi_{3,1} + (0.7)\phi_{4,1} = 4.75091$$

$$i = 4, (0.7)\phi_{3,1} + (4.6)\phi_{4,1} + (0.7)\phi_{5,1} = 5.58710$$

$$i = 5, (0.7)\phi_{4,1} + (4.6)\phi_{5,1} + (0.7)\phi_{6,1} = 5.87261$$

$$i = 6, (0.7)\phi_{5,1} + (4.6)\phi_{6,1} + (0.7)\phi_{7,1} = 5.58715$$

$$i = 7, (0.7)\phi_{6,1} + (4.6)\phi_{7,1} + (0.7)\phi_{8,1} = 4.75091$$

$$i = 8, (0.7)\phi_{7,1} + (4.6)\phi_{8,1} + (0.7)\phi_{9,1} = 3.45158$$

$$i = 9, (0.7)\phi_{8,1} + (4.6)\phi_{9,1} + (0.7)\phi_{10,1} = 1.81461$$

Solving above system of nine equations in nine unknowns with the help of Mat lab, we get the solution as shown in table (4.1.2). These solutions are compared with the exact solutions. From table (4.1.2) it is clear that, the spline solutions are fairly agree with exact solutions. Similarly, for  $j = 1$ , we get another nine algebraic equations. This can be solved by above method.

**Table: 4.1.2:****Solution of One Dimensional Burger's equation using Cubic Spline implicit scheme**

<b>u(x,t)</b>			
<b>x</b>	<b>t = 1/1000</b>	<b>t = 2/1000</b>	<b>t = 3/1000</b>
0	0	0	0
0.1	0.307411091950798	0.305841302887021	0.299893894969247
0.2	0.584730988657748	0.581754201917777	0.570454091783653
0.3	0.804813079352355	0.800739916260907	0.785168781578878
0.4	0.946114558868559	0.941357334317828	0.923056576641411
0.5	0.994803474990543	0.989797764234354	0.970549773219262
0.6	0.946114558868559	0.941357334317828	0.923056576641412
0.7	0.804813079352355	0.800739916260907	0.785168781578878
0.8	0.584730988657748	0.581754201917777	0.570454091783653
0.9	0.307411091950798	0.305841302887021	0.299893894969247
1.0	0	0	0

**Table: 4.1.3:****Comparison of Spline explicit solution and implicit solution with Analytic solution**

<b>x</b>	<b>Implicit Solution</b>	<b>Explicit Solution</b>	<b>Analytic Solution</b>
0	0	0	0
0.1	0.307411091950798	0.305940883977901	0.308058
0.2	0.584730988657748	0.581936464088398	0.585962
0.3	0.804813079352355	0.800966259668508	0.806508
0.4	0.946114558868559	0.941592497237569	0.948108
0.5	0.994803474990543	0.990048751381216	0.996899
0.6	0.946114558868559	0.941592497237569	0.948108
0.7	0.804813079352355	0.800966259668508	0.806508
0.8	0.584730988657748	0.581936464088398	0.585962
0.9	0.307411091950798	0.305940883977901	0.308058
1.0	0	0	0

#### **4.1.3 DISCUSSION OF RESULT:**

Table (4.1.3) and (4.1.4) gives the comparison of both the spline solutions namely explicit and implicit with available exact solutions. From the table (4.1.3) it is clear that, the spline solutions are fairly agree with available solutions of one dimensional Burgers' equation.

## 4.2 SPLINE FORMULA TO SOLVE TWO DIMENSIONAL COUPLED VISCOUS BURGERS' EQUATION:

The two dimensional non-linear coupled viscous burger's equation is

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{1}{\text{Re}} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad (4.2.1)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = \frac{1}{\text{Re}} \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)$$

(4.2.2)

With the initial conditions namely,

$$u(x, y, 0) = \psi_1(x, y); (x, y) \in \Omega$$

$$v(x, y, 0) = \psi_2(x, y); (x, y) \in \Omega$$

And Dirichlet boundary conditions:

$$\left. \begin{aligned} u(x, y, t) &= \xi(x, y); (x, y) \in \partial\Omega \\ v(x, y, t) &= \zeta(x, y); (x, y) \in \partial\Omega \end{aligned} \right\}, t \geq 0$$

Where  $\Omega = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$  is the computational domain and  $\partial\Omega$  is its boundary  $u(x, y, t)$  and  $v(x, y, t)$  are the velocity components to be determined.  $\psi_1, \psi_2, \xi$  and  $\zeta$  are known functions,  $\partial u / \partial t$  is unsteady term,  $u \frac{\partial u}{\partial x}$  is a nonlinear convective term,

$\frac{1}{\text{Re}} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$  is the diffusion term and Re is the Reynolds numbers.

So first we reduce this nonlinear equation to linear form using Cole-Hopf transformation.

First we introduce new function  $\phi(x, y, t)$  such that

$$u(x, y, t) = \frac{-2}{\text{Re}} \cdot \frac{1}{\phi} \frac{\partial \phi}{\partial x} \quad \& \quad v(x, y, t) = \frac{-2}{\text{Re}} \cdot \frac{1}{\phi} \frac{\partial \phi}{\partial y}$$

$$\frac{\partial u}{\partial t} = \frac{-2}{\text{Re}} \left( \frac{\phi_{xt} \phi - \phi_x \phi_t}{\phi^2} \right), \quad \frac{\partial u}{\partial x} = \frac{-2}{\text{Re}} \left( \frac{\phi_{xx} \phi - \phi_x^2}{\phi^2} \right), \quad \frac{\partial u}{\partial y} = \frac{-2}{\text{Re}} \left( \frac{\phi_{xy} \phi - \phi_x \phi_y}{\phi^2} \right),$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{-2}{\text{Re}} \left( \frac{\phi_{xxx} \phi - \phi_{xx} \phi_x - 2\phi_{xx} \phi_x \phi^2 - 2\phi \phi_x^3}{\phi^2} \right)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{-2}{\text{Re}} \left( \frac{\phi_{xyy} \phi - \phi_{xy} \phi_y - \phi_{xy} \phi_y \phi^2 + \phi^2 \phi_x \phi_{yy} - 2\phi \phi_x \phi_y^2}{\phi^2} \right)$$

Substitute all the values onto equation (4.2.1), which leads to

$$\phi_{xt} \phi - \phi_x \phi_t = \frac{1}{\text{Re}} (\phi \phi_{xxx} - \phi_x \phi_{xx}) + (\phi \phi_{xyy} - \phi_x \phi_{yy})$$

Dividing by  $\phi^2$  on both side then we get

$$\left(\frac{\phi_t}{\phi}\right)_x = \frac{1}{\text{Re}} \left[ \left(\frac{\phi_{xx}}{\phi}\right)_x + \left(\frac{\phi_{yy}}{\phi}\right)_x \right]$$

Integrating both side w.r.t. x and simplifying

$$\phi_t = \frac{1}{\text{Re}} (\phi_{xx} + \phi_{yy}) + \alpha_1(y, t) \phi \quad (4.2.3)$$

Similarly using equation (4.2.2) we get another equation

$$\phi_t = \frac{1}{\text{Re}} (\phi_{xx} + \phi_{yy}) + \alpha_2(x, t) \phi \quad (4.2.4)$$

Where  $\alpha_1(t)$  and  $\alpha_2(t)$  are an arbitrary functions depending on t.

Combining the equation (4.2.3) and (4.2.4), we can conclude that satisfies the following

equation 
$$\phi_t = \frac{1}{\text{Re}} (\phi_{xx} + \phi_{yy}) + \alpha(t) \phi \quad (4.2.5)$$

Where  $\alpha(t)$  is an arbitrary function depending on t only.

Without loss of generality, we can choose  $\alpha(t) = 0$  in equation (4.2.5) to simplify the discussion and computation.

$$\phi_t = \frac{1}{\text{Re}} (\phi_{xx} + \phi_{yy}) \quad (4.2.6)$$

Now discretizing the left hand side of PDE (4.2.6) by forward difference formula like finite difference and right side by second derivative of cubic spline  $S(x)$ .

$$\frac{\phi_{i,j,k+1} - \phi_{i,j,k}}{\Delta t} = \frac{1}{\text{Re}} (2S''_{i,j,k}) \quad (4.2.7)$$

Substitute values of  $S''_{ij}$  in below equation

$$S''_{i-1,j,k} + 4S''_{i,j,k} + S''_{i+1,j,k} = \left(\frac{6}{h^2}\right) (\phi_{i-1,j,k} - 2\phi_{i,j,k} + \phi_{i+1,j,k})$$

Then we get

$$\frac{(\phi_{i-1,j,k+1} - \phi_{i-1,j,k})}{2(1/\text{Re})\Delta t} + 4 \frac{(\phi_{i,j,k+1} - \phi_{i,j,k})}{2(1/\text{Re})\Delta t} + \frac{(\phi_{i+1,j,k+1} - \phi_{i+1,j,k})}{2(1/\text{Re})\Delta t} = \left(\frac{6}{h^2}\right) [\phi_{i-1,j,k} - 2\phi_{i,j,k} + \phi_{i+1,j,k}]$$

$$\phi_{i-1,j,k+1} + 4\phi_{i,j,k+1} + \phi_{i+1,j,k+1} = \left(1 + \frac{12\Delta t}{h^2 \cdot \text{Re}}\right) \phi_{i-1,j,k} + \left(4 - \frac{24\Delta t}{h^2 \cdot \text{Re}}\right) \phi_{i,j,k} + \left(1 + \frac{12\Delta t}{h^2 \cdot \text{Re}}\right) \phi_{i+1,j,k} \quad (4.2.8)$$

Above formula is known as **cubic spline explicit formula** to solve two dimensional coupled viscous Burger's equation of the form (4.2.1) & (4.2.2).

For Implicit scheme is unconditionally stable In Implicit method the discretization of the differential equation at any mesh point (i, j) is done by replacing time derivative by the forward difference formula as done in explicit scheme and the space derivative is replaced by average of second derivatives of cubic spline S(x) at the (j)<sup>th</sup> and (j+1)<sup>th</sup> level.

$$\frac{\phi_{i,j,k+1} - \phi_{i,j,k}}{2(1/\text{Re})\Delta t} = S''_{i,j,k} + S''_{i,j,k+1} \quad (4.2.9)$$

The values of  $S''_{i,j+1}$  obtained from equation (4.2.9) and with the help of below equations the following relation is obtained.

$$\begin{aligned} S''_{i-1,j,k} + 4S''_{i,j,k} + S''_{i+1,j,k} &= \left(\frac{6}{h^2}\right) (\phi_{i-1,j,k} - 2\phi_{i,j,k} + \phi_{i+1,j,k}) \\ S''_{i-1,j,k+1} + 4S''_{i,j,k+1} + S''_{i+1,j,k+1} &= \left(\frac{6}{h^2}\right) (\phi_{i-1,j,k+1} - 2\phi_{i,j,k+1} + \phi_{i+1,j,k+1}) \\ \left\{ \frac{\phi_{i-1,j,k+1} - \phi_{i-1,j,k}}{2(1/\text{Re})\Delta t} \right\} + 4 \left\{ \frac{\phi_{i,j,k+1} - \phi_{i,j,k}}{2(1/\text{Re})\Delta t} \right\} + \left\{ \frac{\phi_{i+1,j,k+1} - \phi_{i+1,j,k}}{2(1/\text{Re})\Delta t} \right\} &= \left(\frac{6}{h^2}\right) \{ \phi_{i-1,j,k+1} - 2\phi_{i,j,k+1} + \phi_{i+1,j,k+1} \} \\ \left[ 1 - \frac{12\Delta t}{\text{Re} \cdot h^2} \right] \phi_{i-1,j,k+1} + \left[ 4 + \frac{24\Delta t}{\text{Re} \cdot h^2} \right] \phi_{i,j,k+1} + \left[ 1 - \frac{12\Delta t}{\text{Re} \cdot h^2} \right] \phi_{i+1,j,k+1} & \\ = \left[ 1 + \frac{12\Delta t}{\text{Re} \cdot h^2} \right] \phi_{i-1,j,k} + \left[ 4 - \frac{24\Delta t}{\text{Re} \cdot h^2} \right] \phi_{i,j,k} + \left[ 1 + \frac{12\Delta t}{\text{Re} \cdot h^2} \right] \phi_{i+1,j,k} & \quad (4.2.10) \end{aligned}$$

The above equation (4.2.10) is known as **cubic spline implicit formula** to solve two dimensional coupled viscous Burger's equation of the form (4.2.1) and (4.2.2).

$$\Omega = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$$

$$\phi(x, y, t) = \frac{3}{4} - \frac{1}{4[1 + \exp((-4x + 4y - t)\text{Re}/32)]}; (x, y) \in \Omega \quad (4.2.11)$$

With initial conditions at t = 0

$$\phi(x, y, 0) = \frac{3}{4} - \frac{1}{4[1 + \exp((-4x + 4y)\text{Re}/32)]}; (x, y) \in \Omega \quad (4.2.12)$$

#### 4.2.1 SOLUTIONS WITH SPLINE EXPLICIT METHOD (When $\alpha(t) = 0$ ):

Let the region  $0 \leq x \leq 1$  and  $0 \leq y \leq 1$  be divided into ten sub-intervals, each of length  $h = 0.1$ ,  $Re = 100$ ,  $\Delta t = 0.0001$  then we get

$$\left(1 + \frac{12\Delta t}{Re \cdot h^2}\right) = 1.0012 \quad \& \quad \left(4 - \frac{24\Delta t}{Re \cdot h^2}\right) = 3.9976$$

For  $i = 1, j = 0$  and  $k = 0$

$$i = 1, \phi_{0,0,1} + 4\phi_{1,0,1} + \phi_{2,0,1} = 3.11615$$

$$i = 2, \phi_{1,0,1} + 4\phi_{2,0,1} + \phi_{3,0,1} = 3.35863$$

$$i = 3, \phi_{2,0,1} + 4\phi_{3,0,1} + \phi_{4,0,1} = 3.13342$$

$$i = 4, \phi_{3,0,1} + 4\phi_{4,0,1} + \phi_{5,0,1} = 3.04201$$

$$i = 5, \phi_{4,0,1} + 4\phi_{5,0,1} + \phi_{6,0,1} = 3.01230$$

$$i = 6, \phi_{5,0,1} + 4\phi_{6,0,1} + \phi_{7,0,1} = 3.00330$$

$$i = 7, \phi_{6,0,1} + 4\phi_{7,0,1} + \phi_{8,0,1} = 3.00080$$

$$i = 8, \phi_{7,0,1} + 4\phi_{8,0,1} + \phi_{9,0,1} = 3.00010$$

$$i = 9, \phi_{8,0,1} + 4\phi_{9,0,1} + \phi_{10,0,1} = 3.0000$$

Solving above system of nine equations in nine unknowns with the help of Matlab, we get the solution as shown in table (4.2.1) and plotted in figure. These solutions are compared with the available analytic solutions.

**Table: 4.2.1:****Two dimensional Burger's solution through Cubic Spline explicit scheme**

<b>u(x,t)</b>			
<b>(x, y)</b>	<b>t = 0.5</b>	<b>t = 1</b>	<b>t = 2</b>
(0.1,0.1)	0.5430075040324	0.530075040630	0.50047034687
(0.5,0.1)	0.5003422099447	0.500032209944	0.50000320012
(0.9,0.1)	0.5000032512345	0.500002512345	0.50000012356
(0.3,0.3)	0.5426919817241	0.526919817241	0.50044132454
(0.7,0.3)	0.5003267467823	0.500032674678	0.50000345876
(0.1,0.5)	0.7421563379908	0.721563379908	0.55515123475
(0.5,0.5)	0.5425022937454	0.525022937454	0.50041550021
(0.9,0.5)	0.5003020210944	0.500020210944	0.50001400245
(0.3,0.7)	0.7421264245782	0.731264245782	0.55481189642
(0.7,0.7)	0.5424650406302	0.524650406302	0.50068338564
(0.1,0.9)	0.7499590823444	0.745908234441	0.74421545672
(0.5,0.9)	0.7421317249865	0.713172498652	0.55980200034
(0.9,0.9)	0.5464082344564	0.524082344564	0.51340997821

**4.2.2 SOLUTIONS WITH SPLINE IMPLICIT METHOD (When  $\alpha(t) = 0$ ):**

In similar manner discussed as earlier, using initial and boundary condition, at  $\Delta t = 0.0001$ ,  $Re = 100, h = 0.1$  the solution of equation (4.2.1) is obtained by using cubic spline implicit formula given by equation (4.2.10) as follows.

$$1 - \frac{12\Delta t}{Re h^2} = 0.9999, 4 + \frac{24\Delta t}{Re h^2} = 4.00015$$

$$1 + \frac{12\Delta t}{Re h^2} = 1.00007, 4 - \frac{24\Delta t}{Re h^2} = 3.9998$$

For  $i = 1, j = 0$  and  $k = 0$

$$\begin{aligned}
i = 1, & (0.9999)\phi_{0,0,1} + (4.0015)\phi_{1,0,1} + (0.9999)\phi_{2,0,1} = 3.626589 \\
i = 2, & (0.9999)\phi_{1,0,1} + (4.0015)\phi_{2,0,1} + (0.9999)\phi_{3,0,1} = 3.131408 \\
i = 3, & (0.9999)\phi_{2,0,1} + (4.0015)\phi_{3,0,1} + (0.9999)\phi_{4,0,1} = 3.126608 \\
i = 4, & (0.9999)\phi_{3,0,1} + (4.0015)\phi_{4,0,1} + (0.9999)\phi_{5,0,1} = 3.501582 \\
i = 5, & (0.9999)\phi_{4,0,1} + (4.0015)\phi_{5,0,1} + (0.9999)\phi_{6,0,1} = 3.379726 \\
i = 6, & (0.9999)\phi_{5,0,1} + (4.0015)\phi_{6,0,1} + (0.9999)\phi_{7,0,1} = 3.119773 \\
i = 7, & (0.9999)\phi_{6,0,1} + (4.0015)\phi_{7,0,1} + (0.9999)\phi_{8,0,1} = 3.749889 \\
i = 8, & (0.9999)\phi_{7,0,1} + (4.0015)\phi_{8,0,1} + (0.9999)\phi_{9,0,1} = 3.379725 \\
i = 9, & (0.9999)\phi_{8,0,1} + (4.0015)\phi_{9,0,1} + (0.9999)\phi_{10,0,1} = 4.119779
\end{aligned}$$

Solving above system of nine equations in nine unknowns with the help of Mat lab, we get the solution as shown in table (4.2.2) and plotted in figure. The results are compared with the exact solutions in table (4.2.3). It is clear that, the spline solutions are fairly agree with exact solutions up to ten digits of decimal points. Similarly, for  $j = 1, 2, \dots$  we can get the solutions.

**Table: 4.2.2:**

**Two dimensional Burgers' solution through Cubic Spline implicit scheme**

<b>u(x,t)</b>			
<b>(x,y)</b>	<b>t = 0.5</b>	<b>t = 1</b>	<b>t = 2</b>
(0.1,0.1)	0.530075040630	0.530075040630	0.50047034687
(0.5,0.1)	0.500032209944	0.500032209944	0.50000320012
(0.9,0.1)	0.500002512345	0.500002512345	0.50000012356
(0.3,0.3)	0.526919817241	0.526919817241	0.50044132454
(0.7,0.3)	0.500032674678	0.500032674678	0.50000345876
(0.1,0.5)	0.721563379908	0.721563379908	0.55515123475
(0.5,0.5)	0.525022937454	0.525022937454	0.50041550021
(0.9,0.5)	0.500020210944	0.500020210944	0.50001400245
(0.3,0.7)	0.731264245782	0.731264245782	0.55481189642
(0.7,0.7)	0.524650406302	0.524650406302	0.50068338564
(0.1,0.9)	0.745908234441	0.745908234441	0.74421545672
(0.5,0.9)	0.713172498652	0.713172498652	0.55980200034
(0.9,0.9)	0.524082344564	0.524082344564	0.51340997821

**Table: 4.2.3:****Comparison of Spline explicit solution and implicit solution with Analytic solution**

<b>(x,y)</b>	<b>Implicit Solution</b>	<b>Explicit Solution</b>	<b>Analytic Solution</b>
(0.1,0.1)	0.530075040630	0.5430075040324	0.54332
(0.5,0.1)	0.500032209944	0.5003422099447	0.50035
(0.9,0.1)	0.500002512345	0.5000032512345	0.50000
(0.3,0.3)	0.526919817241	0.5426919817241	0.54332
(0.7,0.3)	0.500032674678	0.5003267467823	0.50035
(0.1,0.5)	0.721563379908	0.7421563379908	0.74221
(0.5,0.5)	0.525022937454	0.5425022937454	0.54332
(0.9,0.5)	0.500020210944	0.5003020210944	0.50035
(0.3,0.7)	0.731264245782	0.7421264245782	0.74221
(0.7,0.7)	0.524650406302	0.5424650406302	0.54332
(0.1,0.9)	0.745908234441	0.7499590823444	0.74995
(0.5,0.9)	0.713172498652	0.7421317249865	0.74221
(0.9,0.9)	0.524082344564	0.5464082344564	0.54332

### 4.2.3 SOLUTIONS WITH SPLINE EXPLICIT AND IMPLICIT METHOD

(When  $\alpha(t) = 1$ ):

In case ready to select  $\alpha(t) = 1$  at that point

$$\phi_t = \frac{1}{\text{Re}}(\phi_{xx} + \phi_{yy}) + \phi$$

$$\phi_t - \phi = \frac{1}{\text{Re}}(\phi_{xx} + \phi_{yy})$$

Presently discretizing cleared out side of over condition by forward difference formula and supplanting right side by the moment subsidiaries of S at  $j^{\text{th}}$  level like express, we get

$$\frac{\phi_{i,j,k+1} - \phi_{i,j,k}}{\Delta t} - \phi_{i,j,k} = \frac{1}{\text{Re}}(2S_{i,j,k}'' )$$

Substitute values of  $S_{ij}''$  in below equation

$$S_{i-1,j,k}'' + 4S_{i,j,k}'' + S_{i+1,j,k}'' = \left(\frac{6}{h^2}\right)(\phi_{i-1,j,k} - 2\phi_{i,j,k} + \phi_{i+1,j,k})$$

Then we get

$$\begin{aligned} \phi_{i-1,j,k+1} + 4\phi_{i,j,k+1} + \phi_{i+1,j,k+1} = & \left(1 + \frac{12\Delta t}{h^2 \cdot \text{Re}} + \Delta t\right)\phi_{i-1,j,k} + \left(4 - \frac{24\Delta t}{h^2 \cdot \text{Re}} + 4\Delta t\right)\phi_{i,j,k} \\ & + \left(1 + \frac{12\Delta t}{h^2 \cdot \text{Re}} + \Delta t\right)\phi_{i+1,j,k} \end{aligned}$$

(4.2.13)

Above formula is known as **cubic spline explicit formula** for  $\alpha(t) = 1$  at to solve three dimensional coupled viscous Burger's equation of the form (4.2.1) and (4.2.2).

$$\frac{\phi_{i,j,k+1} - \phi_{i,j,k}}{\Delta t} - \phi_{i,j,k} = \left(\frac{2}{\text{Re}}\right)(S_{i,j,k}'' + S_{i,j,k+1}'')$$

The values of  $S_{i,j,k+1}''$  obtained from above equation and with the help of below equations the following relation is obtained.

$$S_{i-1,j,k}'' + 4S_{i,j,k}'' + S_{i+1,j,k}'' = \left(\frac{6}{h^2}\right)(\phi_{i-1,j,k} - 2\phi_{i,j,k} + \phi_{i+1,j,k})$$

$$S_{i-1,j,k+1}'' + 4S_{i,j,k+1}'' + S_{i+1,j,k+1}'' = \left(\frac{6}{h^2}\right)(\phi_{i-1,j,k+1} - 2\phi_{i,j,k+1} + \phi_{i+1,j,k+1})$$

$$\begin{aligned}
& \left[1 - \frac{12\Delta t}{\text{Re}.h^2}\right] \phi_{i-1,j,k+1} + \left[4 + \frac{24\Delta t}{\text{Re}.h^2}\right] \phi_{i,j,k+1} + \left[1 - \frac{12\Delta t}{\text{Re}.h^2}\right] \phi_{i+1,j,k+1} \\
& = \left[1 + \frac{12\Delta t}{\text{Re}.h^2} + \Delta t\right] \phi_{i-1,j,k} + \left[4 - \frac{24\Delta t}{\text{Re}.h^2} + 4\Delta t\right] \phi_{i,j,k} + \left[1 + \frac{12\Delta t}{\text{Re}.h^2} + \Delta t\right] \phi_{i+1,j,k}
\end{aligned} \tag{4.2.14}$$

Above formula (4.2.14) is known as **cubic spline implicit formula** for  $\alpha(t) = 1$  at to solve three dimensional coupled viscous Burger's equation of the form (4.2.1) and (4.2.2).

**Table: 4.2.4:**

**Comparison of Spline explicit solution and implicit solution with Analytic solution**

(x,y)	Implicit Solution	Explicit Solution	Analytic Solution
(0.1,0.1)	0.530075040630	0.624836945690	0.54332
(0.5,0.1)	0.500032209944	0.501721849309	0.50035
(0.9,0.1)	0.500002512345	0.499892220331	0.50000
(0.3,0.3)	0.526919817241	0.551724269198	0.54332
(0.7,0.3)	0.500032674678	0.501849309721	0.50035
(0.1,0.5)	0.721563379908	0.760245879110	0.74221
(0.5,0.5)	0.525022937454	0.542502293745	0.54332
(0.9,0.5)	0.500020210944	0.500302021094	0.50035
(0.3,0.7)	0.731264245782	0.742126424578	0.74221
(0.7,0.7)	0.524650406302	0.542465040630	0.54332
(0.1,0.9)	0.745908234441	0.749959082344	0.74995
(0.5,0.9)	0.713172498652	0.742131724986	0.74221
(0.9,0.9)	0.524082344564	0.546408234456	0.54332

**4.2.4 DISCUSSIONS OF RESULTS:**

The obtained solutions are shown in table (4.2.4). An Implicit and explicit Spline Scheme has been proposed for the numerical solutions of two dimensional nonlinear coupled viscous Burgers' equations. The computed results show that the solution obtained by the method is closed to analytic solution. The proposed method is an effective and reliable method for solving a wide range of engineering problems.

### 4.3 SPLINE FORMULA TO SOLVE THREE DIMENSIONAL COUPLED VISCOUS BURGERS' EQUATION:

The Navier-Stokes equation

$$\nabla v = 0 \quad (4.3.1)$$

$$(\rho v)_t + \nabla(\rho v v) + \nabla p - \mu \nabla^2 v = 0 \quad (4.3.2)$$

where  $\rho$  is the density,  $p$  is a pressure,  $v$  is velocity and  $\mu$  is the viscosity of a fluid.

Simplification in (4.3.2) of the x- component of the velocity vector

$$\rho \frac{\partial v^x}{\partial t} + \rho v^x \frac{\partial v^x}{\partial t} + \rho v^y \frac{\partial v^x}{\partial t} + \rho v^z \frac{\partial v^x}{\partial t} + \frac{\partial p}{\partial t} - \mu \left( \frac{\partial^2 v^x}{\partial x^2} + \frac{\partial^2 v^x}{\partial y^2} + \frac{\partial^2 v^x}{\partial z^2} \right) = 0$$

When pressure is absent then above equation simplify

$$\rho \frac{\partial v^x}{\partial t} + \rho v^x \frac{\partial v^x}{\partial t} + \rho v^y \frac{\partial v^x}{\partial t} + \rho v^z \frac{\partial v^x}{\partial t} - \mu \left( \frac{\partial^2 v^x}{\partial x^2} + \frac{\partial^2 v^x}{\partial y^2} + \frac{\partial^2 v^x}{\partial z^2} \right) = 0$$

$$\rho \frac{\partial v^x}{\partial t} + \rho v^x \frac{\partial v^x}{\partial t} + \rho v^y \frac{\partial v^x}{\partial t} + \rho v^z \frac{\partial v^x}{\partial t} = \mu \left( \frac{\partial^2 v^x}{\partial x^2} + \frac{\partial^2 v^x}{\partial y^2} + \frac{\partial^2 v^x}{\partial z^2} \right)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial t} + v \frac{\partial u}{\partial t} + w \frac{\partial u}{\partial t} = \frac{1}{\text{Re}} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \quad (4.3.3)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial t} + w \frac{\partial v}{\partial t} = \frac{1}{\text{Re}} \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) \quad (4.3.4)$$

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial t} + v \frac{\partial w}{\partial t} + w \frac{\partial w}{\partial t} = \frac{1}{\text{Re}} \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) \quad (4.3.5)$$

So first we reduce this nonlinear equation to linear form using Cole-Hopf transformation.

First we introduce new function  $\phi(x, y, t)$  such that

$$u = \frac{-2}{\text{Re}} \cdot \frac{1}{\phi} \frac{\partial \phi}{\partial x}, \quad v = \frac{-2}{\text{Re}} \cdot \frac{1}{\phi} \frac{\partial \phi}{\partial y} \quad \& \quad w = \frac{-2}{\text{Re}} \cdot \frac{1}{\phi} \frac{\partial \phi}{\partial z}$$

$$\frac{\partial u}{\partial t} = \frac{-2}{\text{Re}} \left( \frac{\phi_{xt} \phi - \phi_x \phi_t}{\phi^2} \right), \quad \frac{\partial u}{\partial x} = \frac{-2}{\text{Re}} \left( \frac{\phi_{xx} \phi - \phi_x^2}{\phi^2} \right),$$

$$\frac{\partial u}{\partial y} = \frac{-2}{\text{Re}} \left( \frac{\phi_{xy} \phi - \phi_x \phi_y}{\phi^2} \right), \quad \frac{\partial u}{\partial z} = \frac{-2}{\text{Re}} \left( \frac{\phi_{xz} \phi - \phi_x \phi_z}{\phi^2} \right)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{-2}{\text{Re}} \left( \frac{\phi_{xxx} \phi - \phi_{xx} \phi_x}{\phi^2} - \frac{2\phi_{xx} \phi_x \phi^2 - 2\phi \phi_x^3}{\phi^4} \right)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{-2}{\text{Re}} \left( \frac{\phi_{xyy} \phi - \phi_{xy} \phi_y}{\phi^2} - \frac{\phi_{xy} \phi_y \phi^2 + \phi^2 \phi_x \phi_{yy} - 2\phi \phi_x \phi_y^2}{\phi^4} \right)$$

Substitute all the above values in equation (4.3.3), which leads to

$$\phi_{xt}\phi - \phi_x\phi_t = \frac{1}{\text{Re}} \left[ (\phi\phi_{xxx} - \phi_x\phi_{xx}) + (\phi\phi_{xyy} - \phi_x\phi_{yy}) + (\phi\phi_{xzz} - \phi_x\phi_{zz}) \right] \quad (4.3.6)$$

$$\left( \frac{\phi_t}{\phi} \right)_x = \frac{1}{\text{Re}} \left[ \left( \frac{\phi_{xx}}{\phi} \right)_x + \left( \frac{\phi_{yy}}{\phi} \right)_x + \left( \frac{\phi_{zz}}{\phi} \right)_x \right]$$

$$\phi_t = \frac{1}{\text{Re}} (\phi_{xx} + \phi_{yy} + \phi_{zz}) + \alpha_1\phi \quad (4.3.7)$$

Similarly using equation (4.3.4) and (4.3.5) we get another two equation

$$\phi_t = \frac{1}{\text{Re}} (\phi_{xx} + \phi_{yy} + \phi_{zz}) + \alpha_2\phi \quad (4.3.8)$$

$$\phi_t = \frac{1}{\text{Re}} (\phi_{xx} + \phi_{yy} + \phi_{zz}) + \alpha_3\phi \quad (4.3.9)$$

where  $\alpha_1(t)$ ,  $\alpha_2(t)$  &  $\alpha_3(t)$  are an arbitrary functions depending on t.

Combining equation (4.3.7), (4.3.8) and (4.3.9), we can get

$$\phi_t = \frac{1}{\text{Re}} (\phi_{xx} + \phi_{yy} + \phi_{zz}) + \alpha(t)\phi$$

(4.3.10)

Where  $\alpha(t)$  is an arbitrary function depending on t only

In case ready to select  $\alpha(t) = 0$  in condition (4.3.10) then

$$\phi_t = \frac{1}{\text{Re}} (\phi_{xx} + \phi_{yy} + \phi_{zz}) \quad (4.3.11)$$

In case ready to select  $\alpha(t) = 1$  in condition (4.3.10) then

$$\phi_t = \frac{1}{\text{Re}} (\phi_{xx} + \phi_{yy} + \phi_{zz}) + \phi \quad (4.3.12)$$

The initial and boundary conditions are

$$\phi(x, y, z, 0) = \sin(\pi x) \cdot \cos(\pi y) \cdot \cos(\pi z)$$

$$\phi(0, y, z, t) = \phi(1, y, z, t) = 0$$

Now discretizing the left hand side of PDE (4.3.12) by forward difference formula like finite difference and right side by second derivative of cubic spline  $S(x)$ .

$$\frac{\phi_{1,j,k+1} - \phi_{1,j,k}}{\Delta t} = \frac{1}{\text{Re}} \left( 3S_{1,j,k}'' \right)$$

Substitute values of  $S_{i,j}''$  in below equation

$$S_{i-1,j,k}'' + 4S_{i,j,k}'' + S_{i+1,j,k}'' = \left( \frac{6}{h^2} \right) (\phi_{i-1,j,k} - 2\phi_{i,j,k} + \phi_{i+1,j,k})$$

Then we get

$$\begin{aligned} & \phi_{i-1,j,k+1} + 4\phi_{i,j,k+1} + \phi_{i+1,j,k+1} \\ &= \left( 1 + \frac{18\Delta t}{h^2 \cdot \text{Re}} \right) \phi_{i-1,j,k} + \left( 4 - \frac{36\Delta t}{h^2 \cdot \text{Re}} \right) \phi_{i,j,k} + \left( 1 + \frac{18\Delta t}{h^2 \cdot \text{Re}} \right) \phi_{i+1,j,k} \end{aligned} \quad (4.3.13)$$

Above formula is known as **cubic spline explicit formula** to solve three dimensional coupled viscous Burger's equation of the form (4.3.1) and (4.3.2).

To derive implicit scheme, we discretize the differential equation at any mesh point (i,j), which is done by replacing time derivative by the forward difference formula as done in explicit scheme and the space derivative is replaced by average of second derivatives of cubic spline S(x) at the (j)<sup>th</sup> and (j+1)<sup>th</sup> level.

$$\frac{\phi_{i,j,k+1} - \phi_{i,j,k}}{3(1/\text{Re})\Delta t} = S_{i,j,k}'' + S_{i,j,k+1}'' \quad (4.3.14)$$

The values of  $S_{i,j,k+1}''$  obtained from equation (4.3.14) and with the help of below equations the following relation is obtained.

$$\begin{aligned} & S_{i-1,j,k}'' + 4S_{i,j,k}'' + S_{i+1,j,k}'' = \left( \frac{6}{h^2} \right) (\phi_{i-1,j,k} - 2\phi_{i,j,k} + \phi_{i+1,j,k}) \\ & S_{i-1,j,k+1}'' + 4S_{i,j,k+1}'' + S_{i+1,j,k+1}'' = \left( \frac{6}{h^2} \right) (\phi_{i-1,j,k+1} - 2\phi_{i,j,k+1} + \phi_{i+1,j,k+1}) \\ & \left( 1 - \frac{18\Delta t}{\text{Re} \cdot h^2} \right) \phi_{i-1,j,k+1} + \left( 4 + \frac{36\Delta t}{\text{Re} \cdot h^2} \right) \phi_{i,j,k+1} + \left( 1 - \frac{18\Delta t}{\text{Re} \cdot h^2} \right) \phi_{i+1,j,k+1} \\ &= \left( 1 + \frac{18\Delta t}{\text{Re} \cdot h^2} \right) \phi_{i-1,j,k} + \left( 4 - \frac{36\Delta t}{\text{Re} \cdot h^2} \right) \phi_{i,j,k} + \left( 1 + \frac{18\Delta t}{\text{Re} \cdot h^2} \right) \phi_{i+1,j,k} \end{aligned} \quad (4.3.15)$$

The above equation (4.3.15) is known as **cubic spline implicit formula** to solve three dimensional coupled viscous Burger's equation of the form (4.3.1) and (4.3.2).

### 4.3.1 SOLUTIONS WITH SPLINE EXPLICIT METHOD (IF $\alpha(t) = 0$ ):

Let the region  $0 \leq x \leq 1, 0 \leq y \leq 1$  and  $0 \leq z \leq 1$  be divided into sub-intervals, each of length  $h = 0.25, Re = 100, \Delta t = 0.0001$  then we get

$$\left(1 + \frac{18\Delta t}{Re \cdot h^2}\right) = 1.0002 \quad \& \quad \left(4 - \frac{36\Delta t}{Re \cdot h^2}\right) = 3.9994$$

$$\phi_{i-1,j,k+1} + 4\phi_{i,j,k+1} + \phi_{i+1,j,k+1} = (1.0002)\phi_{i+1,j,k} + (3.9994)\phi_{i,j,k} + (1.0002)\phi_{i-1,j,k}$$

For  $i = 1, j = 0$  and  $k = 0$

$$i = 1, \phi_{0,0,1} + 4\phi_{1,0,1} + \phi_{2,0,1} = 3.828216$$

$$i = 2, \phi_{1,0,1} + 4\phi_{2,0,1} + \phi_{3,0,1} = 5.413918$$

$$i = 3, \phi_{2,0,1} + 4\phi_{3,0,1} + \phi_{4,0,1} = 3.828216$$

$$i = 1, \phi_{0,0,1} + 4\phi_{1,0,1} + \phi_{2,0,1} = 3.828216$$

$$i = 2, \phi_{1,0,1} + 4\phi_{2,0,1} + \phi_{3,0,1} = 5.413918$$

$$i = 3, \phi_{2,0,1} + 4\phi_{3,0,1} + \phi_{4,0,1} = 3.828216$$

$$i = 1, \phi_{0,0,1} + 4\phi_{1,0,1} + \phi_{2,0,1} = 3.828216$$

$$i = 2, \phi_{1,0,1} + 4\phi_{2,0,1} + \phi_{3,0,1} = 5.413918$$

$$i = 3, \phi_{2,0,1} + 4\phi_{3,0,1} + \phi_{4,0,1} = 3.828216$$

Solving above system of equations with the help of Mat lab, we get the solution as shown in table (4.3.1). These solutions are compared with the available solutions. From table (4.3.1) it is clear that, the spline solutions are fairly agree with available solutions.

**Table: 4.3.1:****Three dimensional Burgers' solution through Cubic Spline explicit scheme**

<b>X</b>	<b>Y</b>	<b>Z</b>	<b>Explicit</b>	<b>Analytic</b>
<b>0.25</b>	<b>0</b>	<b>0</b>	0.4484535124	0.4587111814
<b>0.50</b>	<b>0</b>	<b>0</b>	0.8199874564	0.8284494484
<b>0.75</b>	<b>0</b>	<b>0</b>	0.9058749442	0.9127629699
<b>0.25</b>	<b>0.25</b>	<b>0</b>	0.3999456844	0.40217546055
<b>0.50</b>	<b>0.25</b>	<b>0</b>	0.6810045321	0.68859484289
<b>0.75</b>	<b>0.25</b>	<b>0</b>	0.4094587941	0.41170084174
<b>0.25</b>	<b>0.25</b>	<b>0.25</b>	0.3573456875	0.36229483134
<b>0.25</b>	<b>0.50</b>	<b>0.25</b>	0.1399742314	0.14844907905

**4.3.2 SOLUTIONS WITH SPLINE IMPLICIT METHOD:**

In similar manner discussed as earlier, using initial and boundary condition, at  $\Delta t = 0.0001$ ,  $Re = 100, h = 0.25$  the solution of equation (4.2.1) is obtained by using cubic spline implicit formula given by equation (4.3.11) as follows.

$$1 - \frac{18\Delta t}{Re h^2} = 0.9998, \quad 4 + \frac{36\Delta t}{Re h^2} = 4.0005$$

$$1 + \frac{18\Delta t}{Re h^2} = 1.0002, \quad 4 - \frac{36\Delta t}{Re h^2} = 3.9994$$

For  $i = 1, j = 0$  and  $k = 0$

$$i = 1, (0.9998)u_{0,0,1} + (4.0005)u_{1,0,1} + (0.9998)u_{2,0,1} = 3.828216$$

$$i = 2, (0.9998)u_{1,0,1} + (4.0005)u_{2,0,1} + (0.9998)u_{3,0,1} = 5.413918$$

$$i = 3, (0.9998)u_{2,0,1} + (4.0005)u_{3,0,1} + (0.9998)u_{4,0,1} = 3.828216$$

$$i = 1, (0.9998)u_{0,0,1} + (4.0005)u_{1,0,1} + (0.9998)u_{2,0,1} = 3.828216$$

$$i = 2, (0.9998)u_{1,0,1} + (4.0005)u_{2,0,1} + (0.9998)u_{3,0,1} = 5.413918$$

$$i = 3, (0.9998)u_{2,0,1} + (4.0005)u_{3,0,1} + (0.9998)u_{4,0,1} = 3.828216$$

$$i = 1, (0.9998)u_{0,0,1} + (4.0005)u_{1,0,1} + (0.9998)u_{2,0,1} = 3.828216$$

$$i = 2, (0.9998)u_{1,0,1} + (4.0005)u_{2,0,1} + (0.9998)u_{3,0,1} = 5.413918$$

$$i = 3, (0.9998)u_{2,0,1} + (4.0005)u_{3,0,1} + (0.9998)u_{4,0,1} = 3.828216$$

Solving above system of equations with the help of Matlab, we get the solution as shown in table (4.3.2) and plotted in figure. These solutions are compared with the available solutions. From table (4.3.2) it is clear that, the spline solutions are fairly agree with exact solutions.

**Table: 4.3.2:****Three dimensional Burgers' solution through Cubic Spline implicit scheme**

<b>X</b>	<b>Y</b>	<b>Z</b>	<b>Implicit</b>	<b>Analytic</b>
<b>0.25</b>	<b>0</b>	<b>0</b>	0.4584535124	0.4587111814
<b>0.50</b>	<b>0</b>	<b>0</b>	0.8249874564	0.8284494484
<b>0.75</b>	<b>0</b>	<b>0</b>	0.9108749442	0.9127629699
<b>0.25</b>	<b>0.25</b>	<b>0</b>	0.4009945684	0.40217546055
<b>0.50</b>	<b>0.25</b>	<b>0</b>	0.6821045321	0.68859484289
<b>0.75</b>	<b>0.25</b>	<b>0</b>	0.4109458794	0.41170084174
<b>0.25</b>	<b>0.25</b>	<b>0.25</b>	0.3598456875	0.36229483134
<b>0.25</b>	<b>0.50</b>	<b>0.25</b>	0.1429974231	0.14844907905

**Table: 4.3.3:****Comparison of Spline explicit solution and implicit solution with Analytic solution**

<b>X</b>	<b>Y</b>	<b>Z</b>	<b>Implicit</b>	<b>Explicit</b>	<b>Analytic</b>
<b>0.25</b>	<b>0</b>	<b>0</b>	0.4584535124	0.4484535124	0.4587111814
<b>0.50</b>	<b>0</b>	<b>0</b>	0.8249874564	0.8199874564	0.8284494484
<b>0.75</b>	<b>0</b>	<b>0</b>	0.9108749442	0.9058749442	0.9127629699
<b>0.25</b>	<b>0.25</b>	<b>0</b>	0.4009945684	0.3999456844	0.40217546055
<b>0.50</b>	<b>0.25</b>	<b>0</b>	0.6821045321	0.6810045321	0.68859484289
<b>0.75</b>	<b>0.25</b>	<b>0</b>	0.4109458794	0.4094587941	0.41170084174
<b>0.25</b>	<b>0.25</b>	<b>0.25</b>	0.3598456875	0.3573456875	0.36229483134
<b>0.25</b>	<b>0.50</b>	<b>0.25</b>	0.1429974231	0.1399742314	0.14844907905

**4.3.3 DISCUSSIONS OF RESULTS:**

The obtained solutions are shown in table (4.3.3). An Implicit and explicit Spline Scheme has been proposed for the numerical solutions of three dimensional nonlinear coupled viscous Burgers' equations. The computed results show that the solution obtained by the method is closed to analytic solution. The proposed method is an effective and reliable method for solving a wide range of engineering problems.

#### **4.4 CONCLUSION:**

In this chapter, approximate solution for one, two and three dimensional Burgers' equation with a special set of initial condition and boundary condition is derived using Spline explicit and implicit scheme based on Cole-Hopf transformation. The Cole-Hopf transformation turning the strongly nonlinear Burgers' equation into the linear heat equation plays an important role. The accuracy of the numerical solutions indicates that the method is well suited for the solution of one, two and three dimensional Burgers' equation.

**CHAPTER - 5**  
**APPLICATION OF SPLINE COLLOCATION METHOD TO THE**  
**NAVIER STOKES EQUATIONS OF ONE DIMENSIONAL**  
**AS WELL AS TWO DIMENSIONAL**

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In this chapter, we cover a numerical solution of one and two dimensional Navier-Stokes equation (1D & 2D NSE) using Orłowski and Sobczyk Transformation (OST) and Cole-Hopf transformation (CHT). The Navier-Stokes equation is an important governing equation in fluid dynamics which describes the motion of fluid. Applying OST we have reduced 1D & 2D NSE to viscous 1D & 2D Burgers equations, respectively and then solved viscous Burgers equations numerically by using Spline explicit and implicit method based on Cole-Hopf transformation. The effects of pressure gradient and Reynolds number for different cases are studied. We have compared the solution of 1D & 2D NSE including and excluding pressure gradient.

### 5.1 SPLINE FORMULA TO SOLVE ONE DIMENSIONAL NAVIER-STOKES EQUATION:

The one dimensional Navier-Stokes equation (1D NSE) is

$$\frac{\partial u^*}{\partial t^*} + u^* \frac{\partial u^*}{\partial x^*} + \frac{-1}{\rho} \frac{\partial p^*}{\partial x^*} = \left( \frac{1}{\text{Re}} \right) \left( \frac{\partial^2 u^*}{\partial x^{*2}} \right)$$

Using the dimensionless variables

$$x = \frac{x^*}{L}, u = \frac{u^*}{v}, t = \frac{t^* v}{L}, p = \frac{p^*}{\rho v^2}$$

$$\frac{\partial}{\partial t} u(x, t) + u(x, t) \frac{\partial}{\partial x} u(x, t) + P_x = \left( \frac{1}{\text{Re}} \right) \frac{\partial^2}{\partial x^2} u(x, t) \quad (5.1.1)$$

where  $u(x, t)$  denotes velocity, subscript denotes partial differentiations.

We consider pressure gradient as an exponentially decreasing function of  $t$  of the form.

$$-P_x = f(t) = e^{-at-b}$$

$$u_t + uu_x - cu_{xx} = f(t) \quad (5.1.2)$$

where  $f(t) = e^{-at-b}$  and  $c = (1/\text{Re})$

The OST defined as

$$x' = x - \phi(t), t' = t, u' = u - w(t)$$

$$\text{with } w(t) = \int_0^t f(\tau) d\tau = \frac{1}{a} (e^{-b} - e^{-at-b})$$

$$\phi(t) = \int_0^t w(\tau) d\tau = \frac{1}{a} \left( te^{-t} + \frac{e^{-at-b}}{a} - \frac{e^{-b}}{a} \right)$$

Thus we get

$$x' = x - \frac{1}{a} \left( te^{-t} + \frac{e^{-at-b}}{a} - \frac{e^{-b}}{a} \right), t' = t$$

$$u' = u - \frac{1}{a} (e^{-b} - e^{-at-b})$$

$$\frac{\partial u}{\partial t} = \frac{\partial u'}{\partial t'} \left( -\frac{1}{a} e^{-b} + \frac{1}{a} e^{-at-b} \right) + \frac{\partial u'}{\partial t'} + e^{-at-b}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u'}{\partial x'} \quad \text{And} \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u'}{\partial x'^2}$$

Substituting these transformed derivatives in (5.1.2), we get

$$\frac{\partial u'}{\partial t'} + e^{-at-b} + u' \frac{\partial u'}{\partial x'} - c \frac{\partial^2 u'}{\partial x'^2} = e^{-at-b}$$

$$\frac{\partial u'}{\partial t'} + u' \frac{\partial u'}{\partial x'} - c \frac{\partial^2 u'}{\partial x'^2} = 0$$

$$\frac{\partial u'}{\partial t'} + u' \frac{\partial u'}{\partial x'} = c \frac{\partial^2 u'}{\partial x'^2} \quad (5.1.3)$$

Equation (5.1.3) can be linearized by the Cole-Hopf transformation

$$\left( \phi_t + \frac{1}{2} \phi_{x'}^2 \right)_{x'} = (c \phi_{x'x'})_{x'}$$

Now integrating the transformed equation w.r.t.  $x'$ , then we have

$$\phi_t + \frac{1}{2} \phi_{x'}^2 = c \phi_{x'x'} \quad (5.1.4)$$

Then we use the transformation

$$\phi = -2c \ln u$$

$$\phi_t = \frac{-2c}{u} u_t, \quad \phi_{x'} = \frac{-2c}{u} u_{x'}$$

$$\phi_{x'x'} = -2c \left( \frac{u u_{x'x'} - u_{x'} u_{x'}}{u^2} \right)$$

Substitute these derivatives in (5.1.4), we obtain

$$u_t = c u_{x'x'} \quad (5.1.5)$$

$$u(x', 0) = \sin(\pi x') \text{ in } (0, 1)$$

$$u(0, t) = u(1, t) = 0, \quad t' > 0$$

Now discretizing left side of equation (5.1.5) by forward difference formula and replacing right side for by the second derivative “S” at  $j^{\text{th}}$  level like explicit scheme in finite difference, we get

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} = c S''_{i,j} \quad (5.1.6)$$

where  $S''_{i,j}$  denotes  $S''(x_i)$  at  $(i, j)^{\text{th}}$  level.

Substitute values of  $S''_{i,j}$  in below equation

$$S''_{i-1,j} + 4S''_{i,j} + S''_{i+1,j} = \left(\frac{6}{h^2}\right)(u_{i-1,j} - 2u_{i,j} + u_{i+1,j})$$

Then we get

$$\frac{(u_{i-1,j+1} - u_{i-1,j})}{c \Delta t} + 4 \frac{(u_{i,j+1} - u_{i,j})}{c \Delta t} + \frac{(u_{i+1,j+1} - u_{i+1,j})}{c \Delta t} = \left(\frac{6}{h^2}\right)[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}]$$

$$u_{i-1,j+1} + 4u_{i,j+1} + u_{i+1,j+1} = \left(1 + 6\frac{\Delta t}{\text{Re } h^2}\right)u_{i-1,j} + \left(4 - 12\frac{\Delta t}{\text{Re } h^2}\right)u_{i,j} + \left(1 + 6\frac{\Delta t}{\text{Re } h^2}\right)u_{i+1,j} \quad (5.1.7)$$

Above formula (5.1.7) is known as **cubic spline explicit formula** to solve one dimensional Navier-Stokes equation of the form (5.1.1).

For implicit scheme for one dimensional Navier-Stokes equations. Implicit scheme is unconditionally stable In Implicit method the discretization of the differential equation at any mesh point  $(i, j)$  is done by replacing time derivative by the forward difference formula as done in explicit scheme and the space derivative is replaced by average of second derivatives of cubic spline  $S(x)$  at the  $(j)^{\text{th}}$  and  $(j+1)^{\text{th}}$  level.

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} = \left(\frac{c}{2}\right)(S''_{i,j} + S''_{i,j+1}) \quad (5.1.8)$$

The values of  $S''_{i,j+1}$  obtained from equation (3.1.7) and with the help of below equations the following relation is obtained.

$$S''_{i-1,j} + 4S''_{i,j} + S''_{i+1,j} = \left(\frac{6}{h^2}\right)(u_{i-1,j} - 2u_{i,j} + u_{i+1,j})$$

$$S''_{i-1,j+1} + 4S''_{i,j+1} + S''_{i+1,j+1} = \left(\frac{6}{h^2}\right)(u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1})$$

$$\begin{aligned}
& \left(1-3\frac{\Delta t}{\text{Re } h^2}\right)u_{i-1,j+1} + \left(4+6\frac{\Delta t}{\text{Re } h^2}\right)u_{i,j+1} + \left(1-3\frac{\Delta t}{\text{Re } h^2}\right)u_{i+1,j+1} \\
& = \left(1+3\frac{\Delta t}{\text{Re } h^2}\right)u_{i-1,j} + \left(4-6\frac{\Delta t}{\text{Re } h^2}\right)u_{i,j} + \left(1+3\frac{\Delta t}{\text{Re } h^2}\right)u_{i+1,j}
\end{aligned}
\tag{5.1.9}$$

The above equation (5.1.9) is known as **cubic Spline implicit formula** to solve one dimensional Navier Stokes equation of the form (5.1.1).

### 5.1.1 SOLUTIONS WITH SPLINE EXPLICIT SCHEME:

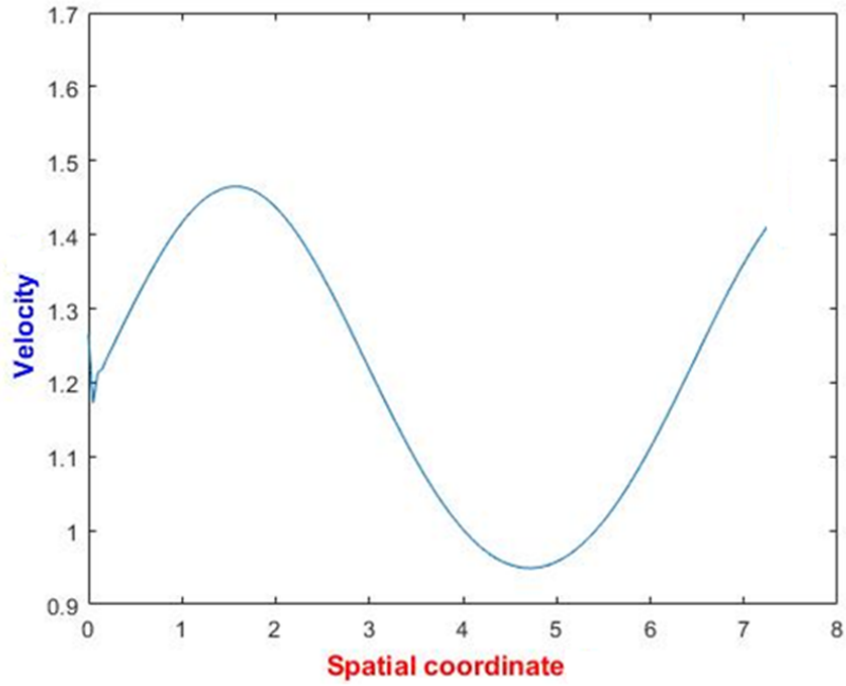
Let the  $c = 1$  and divide the region with  $h = 0.1$  and  $\Delta t = (1/1000) = 0.001$ ,  $\text{Re} = 1$  then  $r = 1.06$

$$1 + (6\Delta t/h^2) = 1.6 \quad \& \quad 4 - (12\Delta t/h^2) = 2.8$$

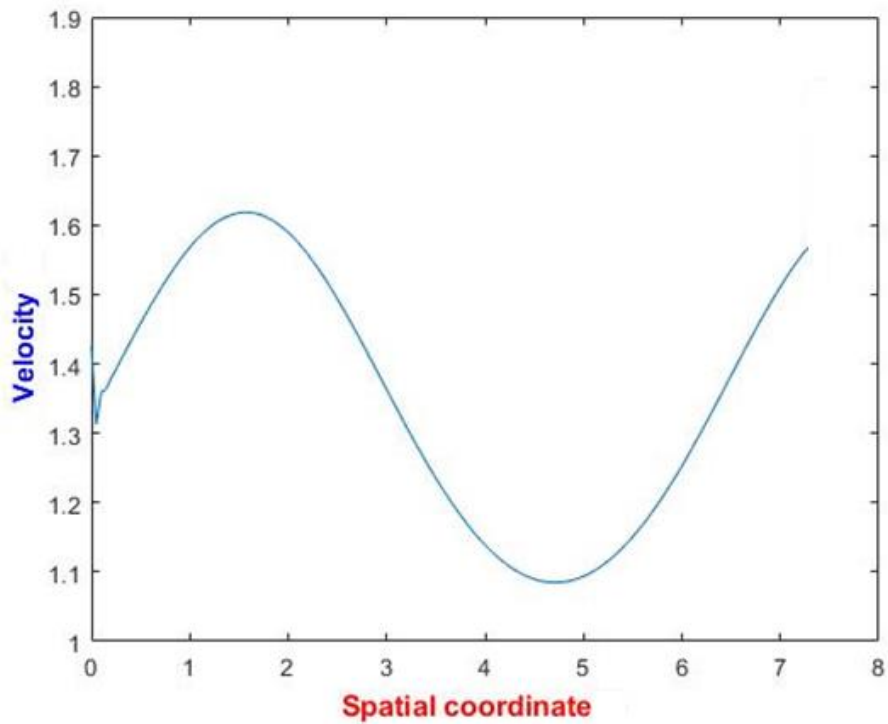
Substitute the values of  $[1 + (6\Delta t/h^2)]$  and  $[4 - (12\Delta t/h^2)]$  in equation (4.1.6) and using initial conditions, we get for  $j = 0, i = 1$

$$\begin{aligned}
i = 1, u_{0,1} + 4u_{1,1} + u_{2,1} &= [(1.6)u_{0,0} + (2.8)u_{1,0} + (1.6)u_{2,0}] = 1.80570 \\
i = 2, u_{1,1} + 4u_{2,1} + u_{3,1} &= 3.434653 \\
i = 3, u_{2,1} + 4u_{3,1} + u_{4,1} &= 4.727394 \\
i = 4, u_{3,1} + 4u_{4,1} + u_{5,1} &= 5.557385 \\
i = 5, u_{4,1} + 4u_{5,1} + u_{6,1} &= 5.843380 \\
i = 6, u_{5,1} + 4u_{6,1} + u_{7,1} &= 5.557385 \\
i = 7, u_{6,1} + 4u_{7,1} + u_{8,1} &= 4.727394 \\
i = 8, u_{7,1} + 4u_{8,1} + u_{9,1} &= 3.434653 \\
i = 9, u_{8,1} + 4u_{9,1} + u_{10,1} &= 1.805703
\end{aligned}$$

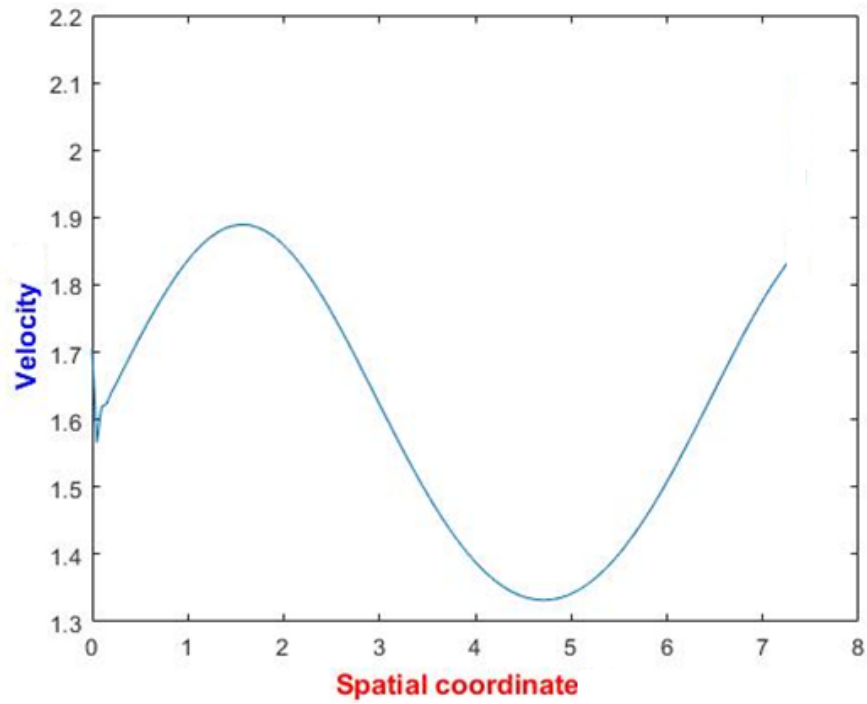
Solving above system of nine equations in nine unknowns with the help of Mat lab, we get the solution as shown in figure (5.1.1) to (5.1.4). These solutions are compared with the available solutions.



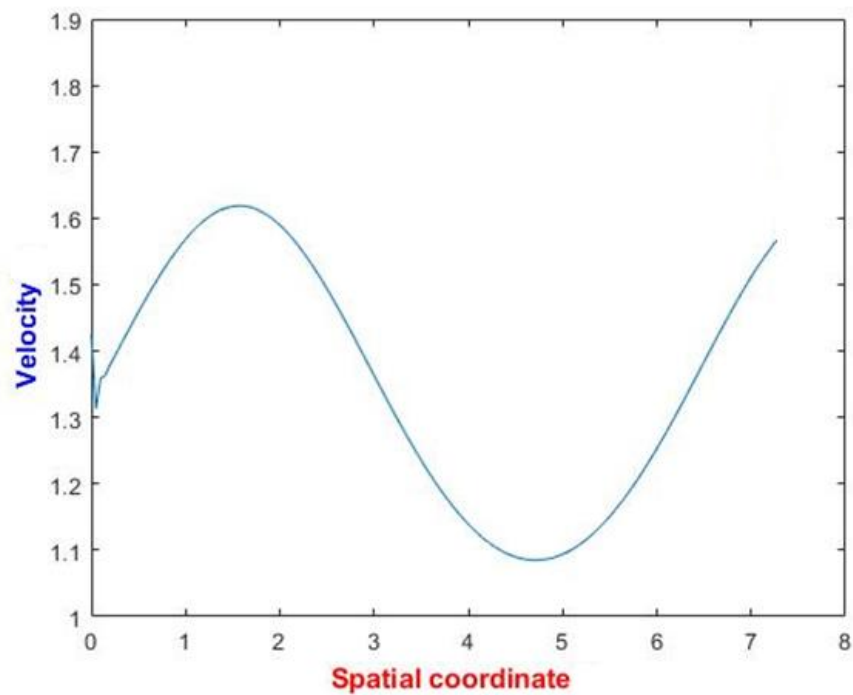
**Figure : 5.1.1 :**  
**Solution of 1D NSE using Spline explicit scheme**  
**(a = 0.8 , b = 0 and c = 0.2)**



**Figure: 5.1.2:**  
**Solution of 1D NSE using Spline explicit scheme**  
**(a = 0.7, b = 0 and c = 0.2)**



**Figure: 5.1.3:**  
**Solution of 1D NSE using Spline explicit scheme**  
**(a = 0.6, b = 0 and c = 0.2)**



**Figure: 5.1.4:**  
**Solution of 1D NSE using Spline explicit scheme at**  
**(a = 0.5, b = 0.3 and c = 0.2)**

### 5.1.2 SOLUTIONS WITH SPLINE IMPLICIT METHOD:

Using the same initial and boundary conditions the solution of equation (5.1.1) is obtained by implicit formula that is given by the equation (5.1.9), derived as follows.

For  $i = 1, j = 0$

Let  $h = 0.1$  and  $\Delta t = (1/1000) = 0.001, c = 1$  then

$$1 - (3\Delta t/h^2) = 0.7 \text{ and } 4 + (6\Delta t/h^2) = 4.6$$

$$1 + (3\Delta t/h^2) = 1.3 \text{ and } 4 - (6\Delta t/h^2) = 3.4$$

$$i = 1, (0.7)u_{0,1} + (4.6)u_{1,1} + (0.7)u_{2,1} = 1.81461$$

$$i = 2, (0.7)u_{1,1} + (4.6)u_{2,1} + (0.7)u_{3,1} = 3.45158$$

$$i = 3, (0.7)u_{2,1} + (4.6)u_{3,1} + (0.7)u_{4,1} = 4.75091$$

$$i = 4, (0.7)u_{3,1} + (4.6)u_{4,1} + (0.7)u_{5,1} = 5.58710$$

$$i = 5, (0.7)u_{4,1} + (4.6)u_{5,1} + (0.7)u_{6,1} = 5.8726$$

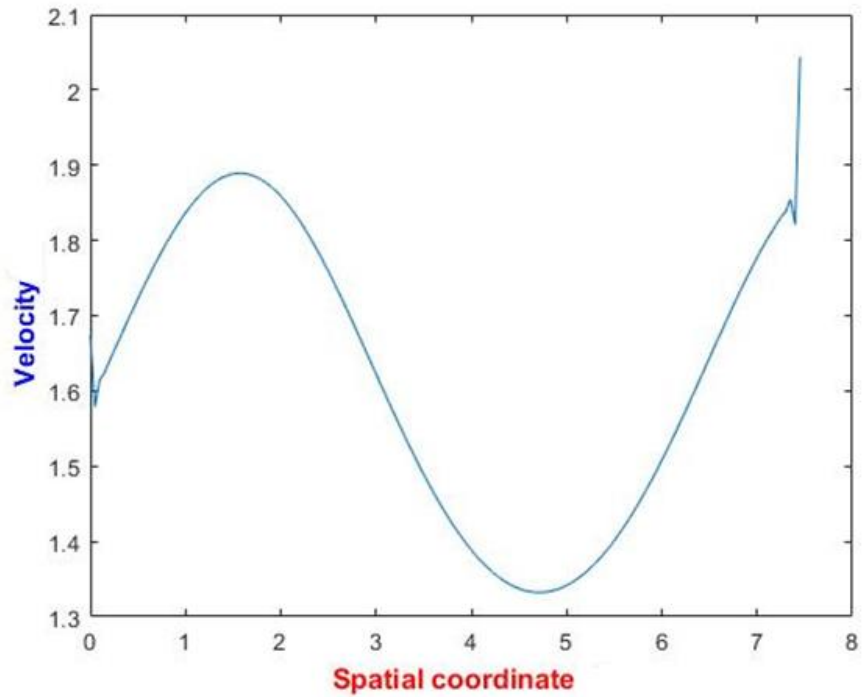
$$i = 6, (0.7)u_{5,1} + (4.6)u_{6,1} + (0.7)u_{7,1} = 5.5871$$

$$i = 7, (0.7)u_{6,1} + (4.6)u_{7,1} + (0.7)u_{8,1} = 4.75091$$

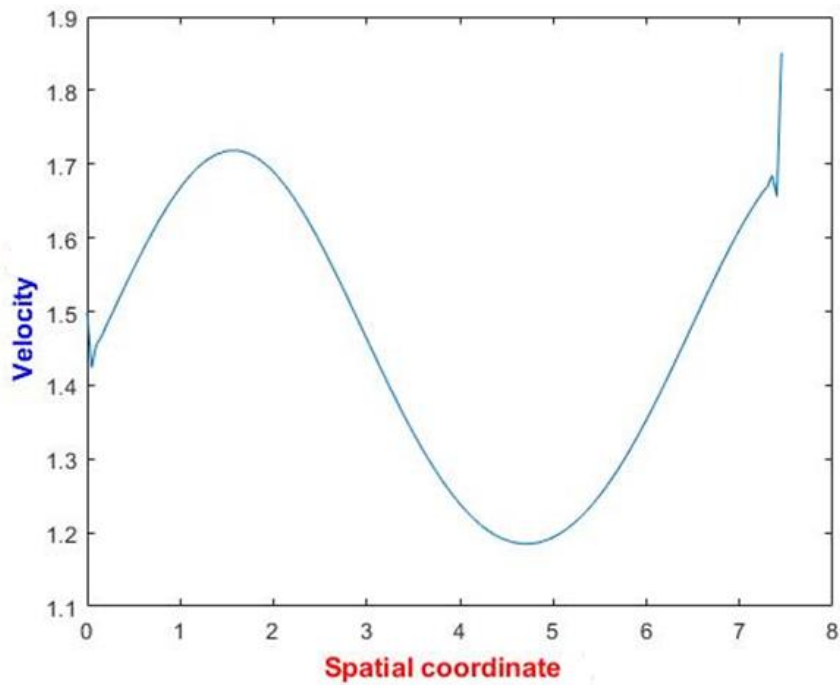
$$i = 8, (0.7)u_{7,1} + (4.6)u_{8,1} + (0.7)u_{9,1} = 3.45158$$

$$i = 9, (0.7)u_{8,1} + (4.6)u_{9,1} + (0.7)u_{10,1} = 1.81461$$

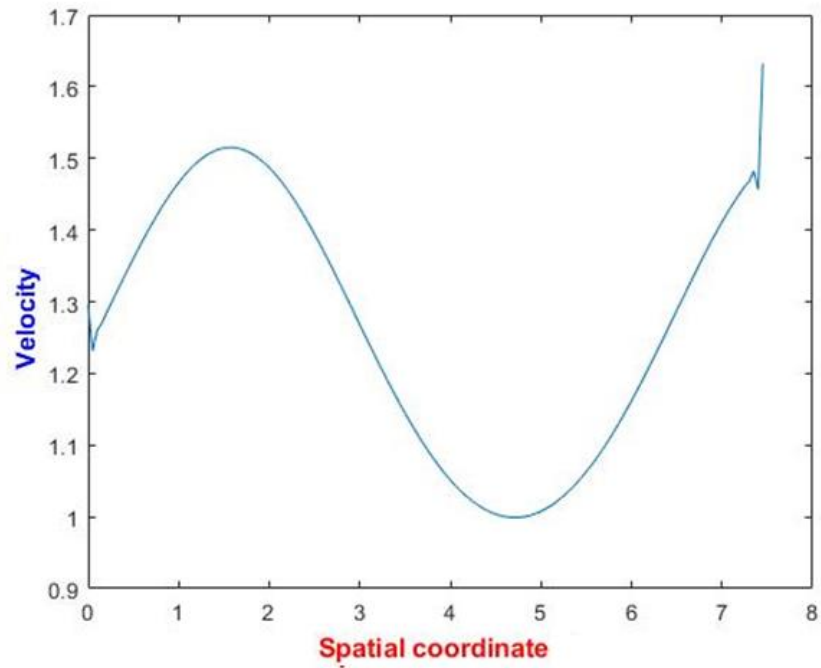
Solving above system of nine equations in nine unknowns with the help of Mat lab, we get the solution as shown in figure (5.1.5) to (5.1.8). These solutions are compared with the available solutions.



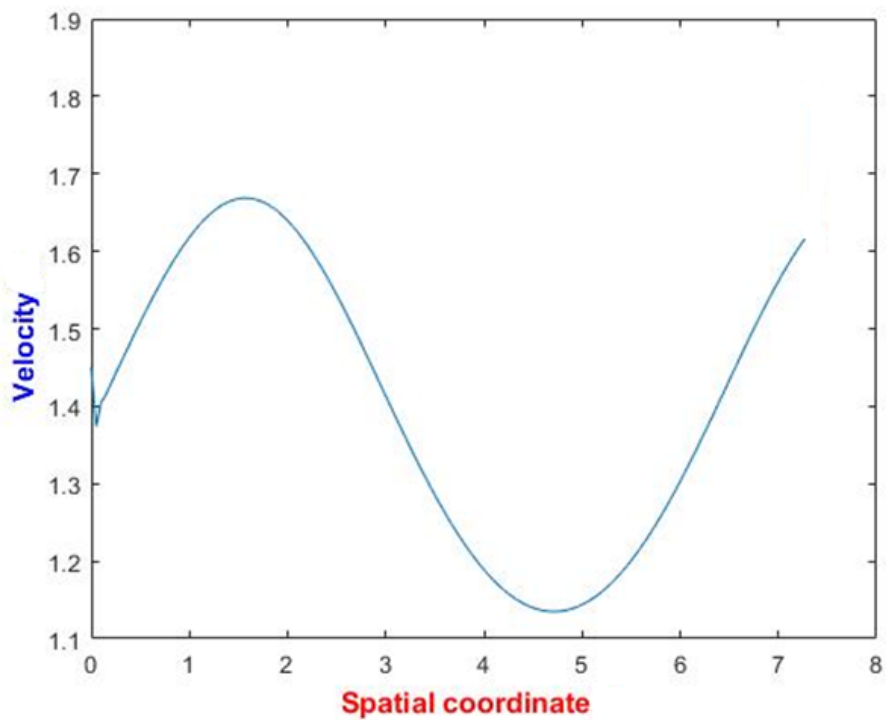
**Figure: 5.1.5:**  
**Solution of 1D NSE using Spline implicit scheme**  
**(a = 0.6, b = 0 and c = 0.2)**



**Figure: 5.1.6:**  
**Solution of 1D NSE using Spline implicit scheme**  
**(a = 0.7, b = 0 and c = 0.2)**



**Figure: 5.1.7:**  
**Solution of 1D NSE using Spline implicit scheme**  
**(a = 0.8, b = 0 and c = 0.2)**



**Figure: 5.1.8:**  
**Solution of 1D NSE using Spline implicit scheme**  
**(a = 0.5, b = 0.3 and c = 0.2)**

### **5.1.3 DISCUSSION OF RESULT:**

We observe that keeping  $b = 0$  and increasing the value of  $a$ , pressure gradient is decreasing continuously in each case velocity is decreasing. Since  $P_x$  is negative pressure gradient decreases in the direction of flow and the pressure gradient is favorable. Here pressure gradient is directly proportional to the velocity. From figure (5.1.1) to (5.1.8) it is clear that, the spline solutions are fairly agree with available solution of one dimensional Navier Stokes equation (1D NSEs)..

## 5.2 SPLINE FORMULA TO SOLVE TWO DIMENSIONAL NAVIER-STOKES

### EQUATION:

The dimensionalised governing equations of the fluid flow are given respectively by the continuity equation [20]

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

x-momentum equation

$$\frac{\partial u^*}{\partial t^*} + u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = \frac{-1}{\rho} \frac{\partial p^*}{\partial x^*} + \nu \left( \frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}} \right) \quad (5.2.1)$$

y-momentum equation

$$\frac{\partial v^*}{\partial t^*} + u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} = \frac{-1}{\rho} \frac{\partial p^*}{\partial y^*} + \nu \left( \frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}} \right) \quad (5.2.2)$$

where  $u$  and  $v$  are the velocity components in the  $x$  and  $y$  directions respectively,  $p$  is the pressure,  $\rho$  is the constant density and  $\nu$  is the kinematic viscosity.

Using the dimensionless variables

$$x = \frac{x^*}{h}, y = \frac{y^*}{h}, u = \frac{u^*}{U}, v = \frac{v^*}{U}, p = \frac{p^*}{\rho U^2}$$

The dimensionalised governing equations are then converted into the non-dimensional form 2D NSEs as

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + P_x = \left( \frac{1}{\text{Re}} \right) \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad (5.2.3)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + P_y = \left( \frac{1}{\text{Re}} \right) \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \quad (5.2.4)$$

where  $\frac{1}{\text{Re}} = \frac{\nu}{Uh}$  And  $\text{Re} = \text{Reynolds number}$

We consider Pressure gradient as an exponentially decreasing function of  $t$  of the form

$$-P_x = f(t) = e^{-at-b} \quad \& \quad -P_y = g(t) = e^{-ct-d}$$

$$u_t + uu_x + vv_y = f(t) + c(u_{xx} + u_{yy}) \quad (5.2.5)$$

$$v_t + uv_x + vv_y = g(t) + c(v_{xx} + v_{yy}) \quad (5.2.6)$$

where  $f(t) = e^{-at-b}$  &  $g(t) = e^{-ct-d}$  And  $c = (1/\text{Re})$

The OST is defined as

$$x' = x - \phi(t), y' = y - \psi(t), t' = t$$

$$u' = u - w(t), v' = v - w'(t)$$

$$w(t) = \int_0^t f(\tau) d\tau = \frac{1}{a} (e^{-b} - e^{-at-b})$$

$$w'(t) = \int_0^t f'(\tau) d\tau = \frac{1}{c} (e^{-d} - e^{-ct-d})$$

$$\phi(t) = \int_0^t w(\tau) d\tau = \frac{1}{a} \left( te^{-b} + \frac{e^{-at-b}}{a} - \frac{e^{-b}}{a} \right)$$

$$\psi(t) = \int_0^t w'(\tau) d\tau = \frac{1}{c} \left( te^{-d} + \frac{e^{-ct-d}}{c} - \frac{e^{-d}}{c} \right)$$

$$x' = x - \frac{1}{a} \left( te^{-t} + \frac{e^{-at-b}}{a} - \frac{e^{-b}}{a} \right), t' = t$$

$$y' = y - \frac{1}{c} \left( te^{-d} + \frac{e^{-ct-d}}{c} - \frac{e^{-d}}{c} \right), t' = t$$

$$u' = u - \frac{1}{a} (e^{-b} - e^{-at-b}) \quad \& \quad v' = v - \frac{1}{c} (e^{-d} - e^{-ct-d})$$

$$\frac{\partial u}{\partial t} = \frac{\partial u'}{\partial x'} \left( -\frac{1}{a} e^{-b} + \frac{1}{a} e^{-at-b} \right) + \frac{\partial u'}{\partial y'} \left( -\frac{1}{c} e^{-d} + \frac{1}{c} e^{-ct-d} \right) + \frac{\partial u'}{\partial t'} + e^{-ct-d}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u'}{\partial x'} \quad \& \quad \frac{\partial u}{\partial y} = \frac{\partial u'}{\partial y'} \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u'}{\partial x'^2} \quad \& \quad \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u'}{\partial y'^2}$$

Substituting these transformed derivatives in (5.3.5), we get

$$\frac{\partial u'}{\partial t'} + e^{-at-b} + u' \frac{\partial u'}{\partial x'} + v' \frac{\partial u'}{\partial y'} - c \left( \frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial y'^2} \right) = e^{-at-b}$$

$$\frac{\partial u'}{\partial t'} + u' \frac{\partial u'}{\partial x'} + v' \frac{\partial u'}{\partial y'} = c \left( \frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial y'^2} \right) \quad (5.2.7)$$

Equation (5.2.7) can be linearized by the Cole-Hopf transformation

$$u' = \psi_{x'}$$

$$\psi_{t'} + \frac{1}{2} \psi_{x'}^2 + \frac{1}{2} \psi_{y'}^2 = c (\psi_{x'x'} + \psi_{y'y'})$$

$$\psi = -2c \ln \phi$$

$$\psi_{t'} = -2c \frac{\phi_{t'}}{\phi}, \psi_{x'} = -2c \frac{\phi_{x'}}{\phi}, \psi_{y'} = -2c \frac{\phi_{y'}}{\phi}$$

$$\psi_{x'x'} = -2c \frac{\phi\phi_{x'x'} - \phi_{x'}\phi_{x'}}{\phi^2} \& \psi_{y'y'} = -2c \frac{\phi\phi_{y'y'} - \phi_{y'}\phi_{y'}}{\phi^2}$$

By using these transformation 2D coupled Burgers' equations can be reduced to 2D diffusion equation

$$\phi_{t'} = c \left( \phi_{x'x'} + \phi_{y'y'} \right) \quad (5.2.8)$$

Which is the well-known second order PDE called heat equation

Now discretizing left side of equation (5.2.8) by forward difference formula and replacing right side for by the second derivative "S" at k<sup>th</sup> level like explicit scheme in finite difference, we get

$$\frac{(\phi_{i,j,k+1} - \phi_{i,j,k})}{\Delta t} = c (2 S''_{i,j,k}) \quad (5.2.9)$$

$$S''_{i-1,j,k} + 4S''_{i,j,k} + S''_{i+1,j,k} = \left( \frac{6}{h^2} \right) (\phi_{i-1,j,k} - 2\phi_{i,j,k} + \phi_{i+1,j,k}) \quad (5.2.10)$$

Then we get

$$\begin{aligned} & \phi_{i-1,j,k+1} + 4\phi_{i,j,k+1} + \phi_{i+1,j,k+1} \\ & = (1 + 12r) \phi_{i-1,j,k} + (4 - 24r) \phi_{i,j,k} + (1 + 12r) \phi_{i+1,j,k} \end{aligned} \quad (5.2.11)$$

where  $r = c \Delta t / h^2$

Equation (5.2.11) is known as **cubic Spline Explicit formula** of 2D NSE.

Same as we discretizing left side of equation (5.2.8) by forward difference formula and replacing right side by the second derivatives of "S" at k<sup>th</sup> and k+1<sup>th</sup> level like implicit scheme in finite difference, we get

$$\frac{(\phi_{i,j,k+1} - \phi_{i,j,k})}{\Delta t} = \frac{c}{2} (S''_{i,j,k} + S''_{i,j,k+1}) \quad (5.2.12)$$

$$S''_{i-1,j,k} + 4S''_{i,j,k} + S''_{i+1,j,k} = \left( \frac{6}{h^2} \right) (\phi_{i-1,j,k} - 2\phi_{i,j,k} + \phi_{i+1,j,k}) \quad (5.2.13)$$

$$S''_{i-1,j,k+1} + 4S''_{i,j,k+1} + S''_{i+1,j,k+1} = \left( \frac{6}{h^2} \right) (\phi_{i-1,j,k+1} - 2\phi_{i,j,k+1} + \phi_{i+1,j,k+1}) \quad (5.2.14)$$

Then we get

$$\begin{aligned}
& (1 - 6r)\phi_{i-1,j,k+1} + (4 + 12r)\phi_{i,j,k+1} + (1 - 6r)\phi_{i+1,j,k+1} \\
& = (1 + 6r)\phi_{i+1,j,k} + (4 - 12r)\phi_{i,j,k} + (1 + 6r)\phi_{i-1,j,k}
\end{aligned} \tag{5.2.15}$$

Equation (5.2.15) is known as Spline Implicit formula of 2D NSE.

$$\Omega = \{(x',y'): 0 \leq x' \leq 1, 0 \leq y' \leq 1\}$$

$$\begin{aligned}
\phi(x',y',t) &= \frac{-4\pi e^{\frac{-5\pi^2 t}{\text{Re}}} \cos(2\pi x') \sin(\pi y')}{\text{Re} \left( 2 + e^{\frac{-5\pi^2 t}{\text{Re}}} \sin(2\pi x') \sin(\pi y') \right)}; (x',y') \in \Omega \\
\psi(x',y',0) &= \frac{-2\pi e^{\frac{-5\pi^2 t}{\text{Re}}} \sin(2\pi x') \cos(\pi y')}{\text{Re} \left( 2 + e^{\frac{-5\pi^2 t}{\text{Re}}} \sin(2\pi x') \sin(\pi y') \right)}; (x',y') \in \Omega
\end{aligned}$$

Initial condition

$$\begin{aligned}
\phi(x',y',0) &= \frac{-4\pi \cos(2\pi x') \sin(\pi y')}{\text{Re}(2 + \sin(2\pi x') \sin(\pi y'))}; (x',y',t) \in \Omega \\
\psi(x',y',0) &= \frac{-2\pi \sin(2\pi x') \cos(\pi y')}{\text{Re}(2 + \sin(2\pi x') \sin(\pi y'))}; (x',y',t) \in \Omega
\end{aligned}$$

Boundary condition

$$\begin{aligned}
\phi(0,y',t) &= \frac{-2\pi e^{\frac{-5\pi^2 t}{\text{Re}}} \sin(\pi y')}{\text{Re}}; (x',y',t) \in \Omega \\
\psi(1,y',t) &= \frac{-2\pi e^{\frac{-5\pi^2 t}{\text{Re}}} \sin(\pi y')}{\text{Re}}; (x',y',t) \in \Omega \\
\phi(x',0,t) &= \phi(x',1,t) = 0; t \geq 0 \\
\psi(0,y',t) &= \psi(1,y',t) = 0; t \geq 0
\end{aligned}$$

### 5.2.1 SOLUTIONS WITH SPLINE EXPLICIT SCHEME:

In equation (5.2.11) we consider  $\text{Re} = 500$  and  $h = 0.05$  then we get

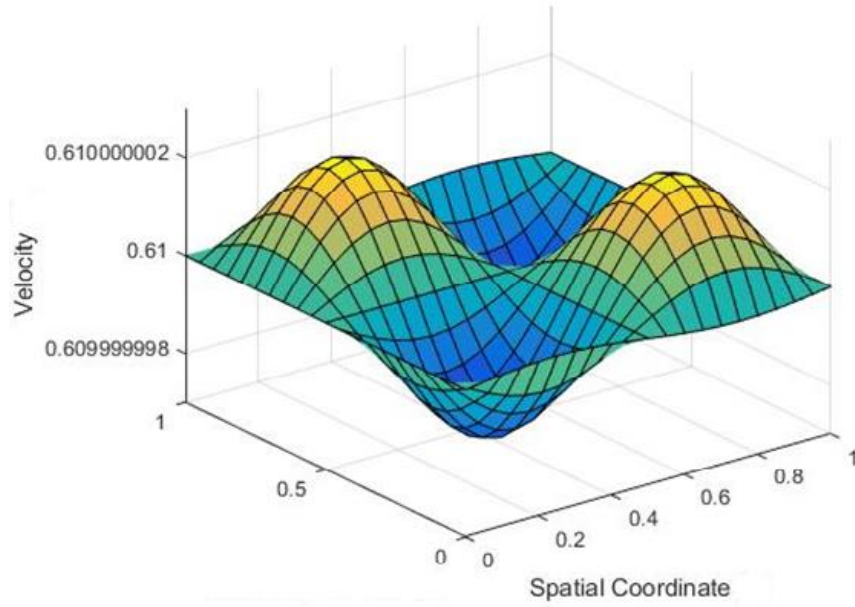
$$\begin{aligned}
1 + (12(0.002)/ (500). (0.05)^2) &= 1.0192 \\
4 - (24(0.002)/ (500). (0.05)^2) &= 3.9616
\end{aligned}$$

For  $i = 1$  to  $9$ ,  $j = 0$  and  $k = 0$  then

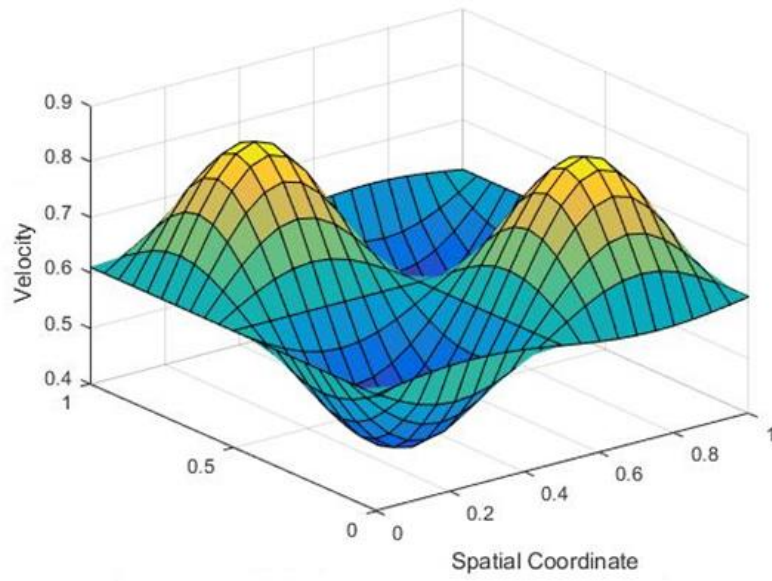
$$\begin{aligned}
\phi_{0,0,1} + 4\phi_{1,0,1} + \phi_{2,0,1} &= -0.0112 \\
\phi_{1,0,1} + 4\phi_{2,0,1} + \phi_{3,0,1} &= -0.0213 \\
\phi_{2,0,1} + 4\phi_{3,0,1} + \phi_{4,0,1} &= -0.0295 \\
\phi_{3,0,1} + 4\phi_{4,0,1} + \phi_{5,0,1} &= -0.0348 \\
\phi_{4,0,1} + 4\phi_{5,0,1} + \phi_{6,0,1} &= -0.0366 \\
\phi_{5,0,1} + 4\phi_{6,0,1} + \phi_{7,0,1} &= -0.0348 \\
\phi_{6,0,1} + 4\phi_{7,0,1} + \phi_{8,0,1} &= -0.0295 \\
\phi_{7,0,1} + 4\phi_{8,0,1} + \phi_{9,0,1} &= -0.0213 \\
\phi_{8,0,1} + 4\phi_{9,0,1} + \phi_{10,0,1} &= -0.0114 \\
\phi_{9,0,1} + 4\phi_{10,0,1} + \phi_{11,0,1} &= -0.0038 \\
\phi_{10,0,1} + 4\phi_{11,0,1} + \phi_{12,0,1} &= 0.0112 \\
\phi_{11,0,1} + 4\phi_{12,0,1} + \phi_{13,0,1} &= 0.0213 \\
\phi_{12,0,1} + 4\phi_{13,0,1} + \phi_{14,0,1} &= 0.0295 \\
\phi_{13,0,1} + 4\phi_{14,0,1} + \phi_{15,0,1} &= 0.0348 \\
\phi_{14,0,1} + 4\phi_{15,0,1} + \phi_{16,0,1} &= 0.0366 \\
\phi_{15,0,1} + 4\phi_{16,0,1} + \phi_{17,0,1} &= 0.0348 \\
\phi_{16,0,1} + 4\phi_{17,0,1} + \phi_{18,0,1} &= 0.0295 \\
\phi_{17,0,1} + 4\phi_{18,0,1} + \phi_{19,0,1} &= 0.0213 \\
\phi_{18,0,1} + 4\phi_{19,0,1} + \phi_{20,0,1} &= 0.0112
\end{aligned}$$

By solving above system of linear equations with the help of Mat lab, we have got the solution of different values of Re and t as shown in below figure (5.2.1) & (5.2.2) and these graphical solutions have been compared with the available graphical solution.

The solution for  $\phi$  is evaluated from 2D Burgers' equations and then by using Inverse OST solutions of NSEs are computed and plotted in figure(5.2.1) and (5.2.2).



**Figure: 5.2.1:**  
**Solution of 2D NSE using Spline explicit scheme**  
**(Re = 500)**



**Figure: 5.2.2:**  
**Solution of 2D NSE using Spline explicit scheme**  
**(Re = 5)**

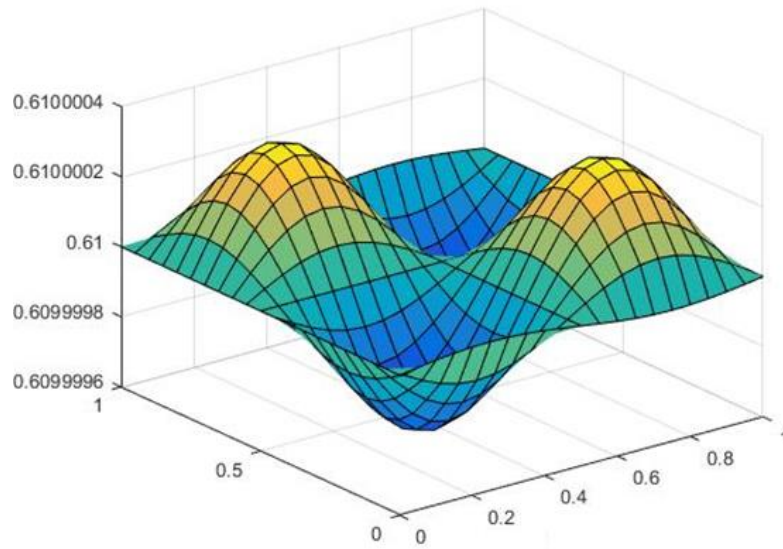
### 5.2.2 SOLUTIONS WITH SPLINE IMPLICIT SCHEME:

In equation (5.2.15) we consider  $Re = 500$  and  $h = 0.05$  then we get

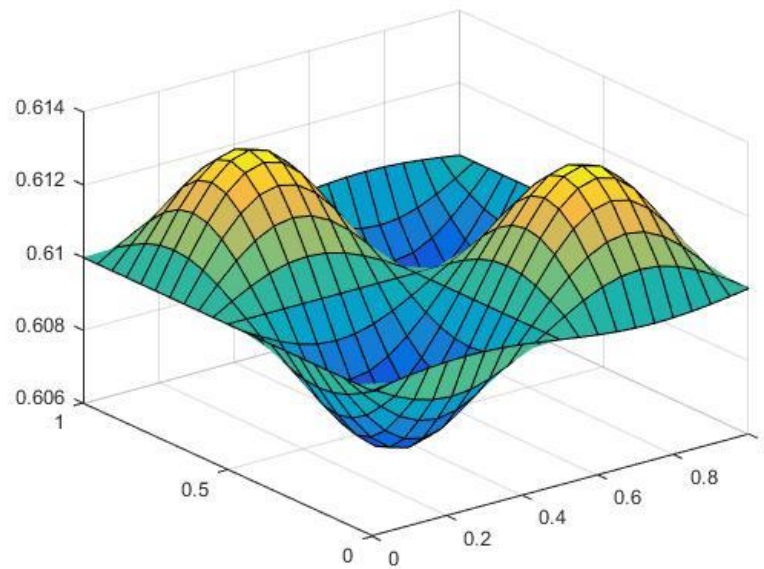
$$\begin{aligned}
 1 - (12(0.002)/ (500). (0.05)^2) &= 0.9808 \\
 1 + (12(0.002)/ (500). (0.05)^2) &= 1.0192 \\
 4 + (24(0.002)/ (500). (0.05)^2) &= 4.0384 \\
 4 - (24(0.002)/ (500). (0.05)^2) &= 3.9616 \\
 (0.9808)\phi_{0,1,1} + (4.0384)\phi_{1,1,1} + (0.9808)\phi_{2,1,1} &= -0.0112 \\
 (0.9808)\phi_{1,1,1} + (4.0384)\phi_{2,1,1} + (0.9808)\phi_{3,1,1} &= -0.0213 \\
 (0.9808)\phi_{2,1,1} + (4.0384)\phi_{3,1,1} + (0.9808)\phi_{4,1,1} &= -0.0295 \\
 (0.9808)\phi_{3,1,1} + (4.0384)\phi_{4,1,1} + (0.9808)\phi_{5,1,1} &= -0.0348 \\
 (0.9808)\phi_{4,1,1} + (4.0384)\phi_{5,1,1} + (0.9808)\phi_{6,1,1} &= -0.0366 \\
 (0.9808)\phi_{5,1,1} + (4.0384)\phi_{6,1,1} + (0.9808)\phi_{7,1,1} &= -0.0348 \\
 (0.9808)\phi_{6,1,1} + (4.0384)\phi_{7,1,1} + (0.9808)\phi_{8,1,1} &= -0.0295 \\
 (0.9808)\phi_{7,1,1} + (4.0384)\phi_{8,1,1} + (0.9808)\phi_{9,1,1} &= -0.0213 \\
 (0.9808)\phi_{8,1,1} + (4.0384)\phi_{9,1,1} + (0.9808)\phi_{10,1,1} &= -0.0112 \\
 (0.9808)\phi_{9,1,1} + (4.0384)\phi_{10,1,1} + (0.9808)\phi_{11,1,1} &= -0.0038 \\
 (0.9808)\phi_{10,1,1} + (4.0384)\phi_{11,1,1} + (0.9808)\phi_{12,1,1} &= 0.0112 \\
 (0.9808)\phi_{11,1,1} + (4.0384)\phi_{12,1,1} + (0.9808)\phi_{13,1,1} &= 0.0213 \\
 (0.9808)\phi_{12,1,1} + (4.0384)\phi_{13,1,1} + (0.9808)\phi_{14,1,1} &= 0.0295 \\
 (0.9808)\phi_{13,1,1} + (4.0384)\phi_{14,1,1} + (0.9808)\phi_{15,1,1} &= 0.0348 \\
 (0.9808)\phi_{14,1,1} + (4.0384)\phi_{15,1,1} + (0.9808)\phi_{16,1,1} &= 0.0366 \\
 (0.9808)\phi_{15,1,1} + (4.0384)\phi_{16,1,1} + (0.9808)\phi_{17,1,1} &= 0.0348 \\
 (0.9808)\phi_{16,1,1} + (4.0384)\phi_{17,1,1} + (0.9808)\phi_{18,1,1} &= 0.0295 \\
 (0.9808)\phi_{17,1,1} + (4.0384)\phi_{18,1,1} + (0.9808)\phi_{19,1,1} &= 0.0213 \\
 (0.9808)\phi_{18,1,1} + (4.0384)\phi_{19,1,1} + (0.9808)\phi_{20,1,1} &= 0.0112
 \end{aligned}$$

By solving above system of linear equations with the help of Matlab, we have got the solution of different values of  $Re$  as shown in below figure (5.2.3) & (5.2.4) and these graphical solutions have been compared with the available solution.

The solution for  $\phi$  is evaluated from 2D Burgers' equations and then by using Inverse OST solutions of NSEs are computed and plotted in figure(5.2.3) and (5.2.4).



**Figure: 5.2.3:**  
**Solution of 2D NSE using Spline Implicit scheme at  $Re = 500$**



**Figure: 5.2.4:**  
**Solution of 2D NSE using Spline Implicit scheme at  $Re = 5$**

### **5.2.3 DISCUSSION OF RESULT:**

In this study, we have shown how to solve numerical solution of 2D NSEs by using OST and CHT with the help of Spline Collocation Method. The method is simple and can be used to find general solution of the governing equations when we set initial and boundary conditions.

From figure (5.2.1) indicates that our solution profile in three different time steps agree with the convection-diffusion type qualitative behavior. It is clear that, the spline solutions are acceptably agree with available solution of two dimensional Navier Stokes equation (2D NSEs).

### **5.3 CONCLUSION:**

We have obtained numerical solution of one dimensional (1D) NSE using the computational results obtained by implementing Mat lab Software we have found important impact of pressure gradient term on the solution of one dimensional (1D) NSE. Reynolds number also plays a vital role on the solution of 1D NSE which may specified as when Reynolds number is low then we get smooth solution.

In this section (5.2) we have shown how to solve numerical solution of two dimensional incompressible Navier-Stokes equation (2D NSEs) for a time dependent exponentially decreasing pressure gradient term using Orłowski and Soczyk transformation(OST) and Cole-Hopf transformation with the help of Spline explicit and implicit scheme. Reynolds number also play a vital role on the numerical solution of 2D NSE. The method is simple and can be used to find general solution of the 2D NSEs when we set initial and boundary conditions.

## **Chapter-6**

### **CONCLUSION AND SCOPE OF FURTHER WORK**

- 6.1 Conclusion
- 6.2 Scope of Future work

## 6.1 Conclusion:

The thesis exhibits the study of Spline Collocation Method with different types of approaches to solve the linear and non-linear boundary value problems in partial differential equations. The various features of the spline collocation technique are expanded the field of numerical solution to the partial differential equations. A special emphasize is given to the applicability and reliability of the method of spline collocation, to see this, the spline functions were used to interpolate the solution to various types of differential equations occurring in the study of several physical phenomena in engineering sciences. In all, it is worthy to mention that the spline collocation technique with all the above approaches are helpful to solve various type of partial differential problems. According to the experience during the research work, it can be concluded that, cubic splines are better for the linear and nonlinear partial differential problems. Beauty of Spline explicit and implicit scheme is no need to convert partial differential problems into ordinary differential form. Coupled equations problem can be solved using this method in less iteration only.

We observed that, Spline implicit scheme gives better approximation in comparison to the explicit scheme. It is also noticed that, as a particular case, if problems are defined with Neumann and Dirichlet boundary conditions, then spline explicit and implicit schemes give better solution as compared to homotopy perturbation method.

According to the experience during the research work, it can be concluded that, Splines schemes show better results for the problems including one, two and three dimensional partial differential equations. The benefit of the Spline collocation method is, nonlinear problems can be solved directly without converting PDE to ODE. More accurate results can be obtained by reducing the mesh size of the interval. The restriction of this method is without using Hofp-Cole and OST transformations we could not solve nonlinear PDEs.

The Present work justifies the applicability and reliability of the spline collocation method. We conclude that spline explicit and implicit schemes are effective and quite encouraging than the finite difference method. It will be possible that same technique can be applied for higher order partial differential equation. Thus the wide applicability of spline collocation is sought in this thesis.

## **6.2 Scope of further work:**

The applications we have examined are few but sufficient enough to illustrate the effectiveness and potentialities of spline explicit and implicit scheme. It is hoped that others will be able to apply the method with successful results and our experience will help them to obtain solutions to increasingly more difficult and complicated problems in engineering sciences, applied sciences and other fields.

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## List of Publications

### Papers Published in Referred Journals

1. Nileshkumar A. Patel, Jigisha U. Pandya. (2016). A Numerical approach for solving nonlinear boundary value problems in finite domain using Spline collocation Method. International Journal for Innovative Research in Science & Technology, 3, 318-321.
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## **Papers Presented in Conferences**

1. Patel Nileshkumar A., & Pandya Jigisha U, “A Numerical Method for the Heat Equation with Dirichlet and Neumann Conditions,” **International conference on research and innovations in science, Engineering & technology**, BVM, Vidyanagar, February – 2017.
2. Patel Nileshkumar A., & Pandya Jigisha U, “Vibrating String Problem Solved Using Spline Collocation Method with Dirichlet and Neumann Conditions,” Incompressible Fluid Past Flat a Plate, **National Conference on Progress, Research and Innovation in Mechanical Engineering**, SCET, Surat, March – 2017.
3. Patel Nileshkumar A., & Pandya Jigisha U, “A Numerical Method for the Heat Equation with Dirichlet and Neumann Conditions,” **International conference on research and innovations in science, Engineering & technology**, BVM, Vidyanagar, February – 2020.